Groups, measures, and the NIP

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Abstract

We discuss measures, invariant measures on definable groups, and genericity, often in an NIP (failure of the independence property) environment. We complete the proof of the third author's conjectures relating definably compact groups G in saturated o-minimal structures to compact Lie groups. We also prove some other structural results about such G, for example the existence of a left invariant finitely additive probability measure on definable subsets of G. We finally introduce a new notion "compact domination" (domination of a definable set by a compact space) and raise some new conjectures in the o-minimal case.

1 Introduction

One of the occasions for writing this paper is the completion of the proof of the "o-minimal group conjectures" of the third author, from [23]. Among the new ingredients are (i) the use of invariant measures on definable sets in the presence of the NIP (failure of the independence property), and (ii) the

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identification of a certain property (finitely satisfiable generics) which can be used in an inductive proof, and is of interest in its own right.

The measures appear in Keisler's paper [13] which is a strong influence on our work. In Keisler's work, the theory of forking is in a sense extended from stable theories to theories without the independence property, but replacing complete types by measures (on the Boolean algebra of definable sets). It is somewhat amusing to note that Keisler's work was roughly contemporaneous with early work on *o*-minimality which was also motivated by the attempt to generalize stability to suitable ordered structures.

Our work may also overlap to some extent with recent papers of Shelah on theories without the independence property (for example [26], [27]).

In any case, we take the opportunity in this paper to expand on and develop some theory, not all of which is directed towards the proof of the *o*-minimal group conjectures.

Stability and stable group theory are at the core of "pure" or "abstract" model theory. Recall Shelah's result that T is stable iff T does not have the strict order property and does not have the independence property (see [26]. There has been considerable work on generalizing stability to particularly nice theories without the strict order property, namely the simple theories. So part of this paper is around developing some theory in an "orthogonal" direction, namely for certain theories T without the independence property. Another aspect of this paper is the "model theory of the standard part map".

In Section 2, we recall and elaborate on some of Keisler's notions from [13]. In particular we discuss smooth, definable, and finitely satisfiable measures. In Section 3, we discuss some consequences of NIP, sometimes in the presence of measures. Include here is a "Borel definability" of coheirs assuming NIP. In Section 4, we introduce the "finitely satisfiable generics" property for definable groups G, stating which aspects of stable group theory are valid in this situation. In Section 5 we discuss in general "definably amenable groups", namely groups with a left invariant measure on the definable sets. In Section 6 we prove various results around existence of G^{00} and existence of invariant measures under the NIP assumption. In Section 7 we take a short diversion to explain how our results can generalize to the class of "inductively definable" groups. In Section 8 we prove the full conjecture from [23]:

(*) *If G is a definably compact group definable in a saturated o-minimal expansion of a real closed field, then the quotient G/G^{00} of G by its smallest type-definable subgroup of bounded index G^{00}, is, when equipped with the logic topology, a compact Lie group whose dimension (as a Lie group) equals the*

dimension of G (as a definable set in an o-minimal structure).

The proof rests on and continues a number of earlier papers [23], [2], [4], [19], and [7]. We will give below a guide for the reader who is interested in a fast path to the proof of (*).

In Section 9 and 10, we isolate a new notion, of "compact domination", and conjecture that in fact a definably compact group G in an o-minimal structure is compactly dominated by G/G^{00}. We then prove this in several special cases.

Guide to the proof of (*). The proof is carried out in section 8. Globally it proceeds by induction on $dim(G)$. The two extreme cases are when (a) G is commutative, and (b) G is definably simple. The "new" ingredient for case (a) is use of the amenability of G (namely the existence of an invariant finitely additive measure on all subsets of G) together with the NIP. The key sequence of preliminary results is Lemma 2.8, Proposition 3.3, Corollary 3.4 and Proposition 6.3. Case (a) is proved in Lemma 8.2. Case (b) was proved in [19] under the weaker hypothesis of "G has very good reduction". This is discussed in Lemma 8.3 of the current paper. For the induction step, one may assume G has a normal commutative definable subgroup N. But we need to know more than simply that (*) holds for G/N and N. Namely we require that both G/N and N have the "finitely satisfiable generics" property. The fsg is introduced in section 4, and Proposition 4.2 is crucial. In Cases (a) and (b) we actually prove in addition that the relevant groups have the fsg property. Proposition 4.5 shows that from the fsg for G/N and N we can conclude the fsg for G. An argument using Corollary 4.3 shows that (*) holds for G.

Our notation is standard. We work in a large saturated model $\bar{M}$ of a complete first order, possibly many-sorted theory T in a language L. If we assume that $|\bar{M}| = \bar{\kappa}$ then by a "small" or "bounded" set we mean a set of cardinality $< \bar{\kappa}$. x, y denote finite sequences of variables unless we say otherwise. $A, B, ..$ denote small subsets of $\bar{M}$. $M, N, ..$ denote small elementary substructures of $\bar{M}$. "Type-definable" means the intersection of a small collection of definable sets, and a "bounded type-definable equivalence relation" is a type-definable equivalence relation with a bounded number of classes. We refer to [24] for any background on stability.

T is said to have the NIP (for "not the independence property") if there is no formula $\phi(x, y) \in L$ and $\langle a_i : i < \omega \rangle$ and $\langle b_w : w \subseteq \omega \rangle$ such that

$\models \phi(a_i, b_w)$ iff $i \in \omega$. Stable and o-minimal theories, as well as the theory of the p-adic field are all examples of theories with NIP, while simple unstable theories all have the independence property.

If G is a group definable in $\bar{M}$ then G^{00} is the smallest type-definable subgroup of bounded index in G, if there is such. If E is a type-definable equivalence relation on a definable set X with a bounded number of classes, then the logic topology on X/E is given by: $C \subseteq X/E$ is closed if the pre-image of C in X is type-definable.

In various parts of the paper we will make use of standard facts and techniques regarding indiscernibles, which the referee has asked us to explain. One of these facts is that given a complete theory T, and cardinal μ there is a cardinal λ such that if $\{a_\alpha : \alpha < \lambda\}$ is a set of μ-tuples in some saturated model of T, then there is an indiscernible sequence $(b_i : i < \omega)$ of μ-tuples, such that for every n, $tp(b_0, .., b_{n-1}) = tp(a_{\alpha_0}, ..., a_{\alpha_{n-1}})$ for some $\alpha_0 < .. < \alpha_{n-1} < \lambda$. This is an application of the Erdös-Rado Theorem. A statement and proof appears in [10] (Theorem 1.13) for example. When using this fact we will just say "by Erdös-Rado". Another method is "stretching" indiscernibles: namely given an indiscernible sequence $(a_i : i < \omega)$ we can, for any totally ordered set I, find an indiscernible sequence $(b_i : i \in I)$ such that for each n and $i_0 < ... < i_n$ in I, $tp(b_{i_0}, ..., b_{i_{n-1}}) = tp(a_0, .., a_{n-1})$. This is of course just by compactness.

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2 Definable functions and measures

We consider here functions of one kind or another from sorts, or definable sets in $\bar{M}$, to compact Hausdorff spaces C, such as the closed interval $[0, 1]$.

Definition 2.1. *Let X be an A-definable set in M, C some compact Hausdorff space of bounded size, and f a map from X to C. We will say that f is* definable over A, *if for any closed subset C_1 of C, $f^{-1}(C_1) \subseteq X$ is type-definable over A in M.*

Example 2.2. *(i) The* tautological *map s from X to its Stone space $S_X(A)$: $s(b) = tp(b/A)$. Note that a map f from X to a compact Hausdorff space C will be definable over A just if $f = g \circ s$ with g a continuous map from $S_X(A)$ to C. So the tautological definable map s is also universal.*
(ii) Let A be a small subset of sort X in $\bar{M}$, and $\phi(x, y)$ a formula, with x of sort X and y of sort Y. Identify the power set of A with the compact space $2^{|A|}$. Let $f : Y \to 2^{|A|}$ be given by $f(b) = \{a \in A : \models \phi(a, b)\}$. Then, as is easy to verify, f is definable over A.

In Definition 2.1, note that if $f : X \to C$ is definable, then $f(X) \subseteq C$ is closed (because as in Example 2.2(i), f can be identified with a continuous map between compact spaces hence its image is closed). So we may assume f to be onto.

In fact definable maps as in Definition 2.1 amount to the same thing as quotienting by bounded type-definable equivalence relations:

Remark 2.3. *Let X be definable over A in M.*
(i) Let f be a definable (over A) map from X onto the compact Hausdorff space C in the sense of Definition 2.1. Let $E = \{(x, y) \in X \times X : f(x) = f(y)\}$. Then E is an A-type-definable equivalence relation of bounded index, and f induces a homeomorphism between X/E with the logic topology and the space C.
(ii) Conversely, if E is a bounded A-type-definable equivalence relation on X, X/E is equipped with the logic topology, and M_0 is a small model containing A and a representative for each E-class, then the quotient map $f : X \to X/E$ is an M_0-definable map from X onto the compact Hausdorff space X/E.

Proof. (i) For each pair C_1, C_2 of closed subsets of C such that $C_1 \cup C_2 = C$, let $E_{C_1,C_2} = \{(x, y) \in X \times X$ such that either $f(x) \in C_1$ and $f(y) \in C_1$ or $f(x) \in C_2$ and $f(y) \in C_2\}$. So E_{C_1,C_2} is type-definable over A. As X is Hausdorff, E is the intersection of all E_{C_1,C_2} hence is also type-definable. Identifying X/E with C we see that the logic topology on C refines the original topology on C. As both topologies are compact Hausdorff they agree. E is of bounded index since the pre-image of each singleton in C is

type-definable over a fixed set A.
(ii) If $C \subseteq X/E$ is closed, then by definition $f^{-1}(C)$ is type-definable. But $f^{-1}(C)$ is also M_0-invariant, hence it is type-definable over M_0. $\square$

So Definition 2.1 is cosmetic. However it enables some unification of various notions, as well as some clean statements. For example the conjecture from [23] can now be restated as:

If G is a definably connected definably compact group in a saturated o-minimal structure M then there is a definable surjective homomorphism f from G to a compact Lie group G_1 where $dim(G_1)$ equals the o-minimal dimension of G. Moreover any other definable homomorphism from G into a compact group factors through f.

We now recall the probablity measures on definable sets considered by Keisler [13]. We will call these *Keisler measures*. Let us fix again a sort or definable set X in $\bar{M}$ which we assume to be $\emptyset$-definable. $Def(X)$ will denote the subsets of X definable (with parameters) in $\bar{M}$, and $Def_A(X)$ those sets defined over A. (So we identify $Def(X)$ with $Def_{\bar{M}}(X)$.)

Definition 2.4. (i) A *Keisler measure* μ on X over A is a finitely additive probability measure on $Def_A(X)$; namely a map μ from $Def_A(X)$ to the interval $[0,1]$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$ and for $Y, Z \in Def_A(X)$, $\mu(Y \cup Z) = \mu(Y) + \mu(Z) - \mu(Y \cap Z)$.
(ii) *A (global) Keisler measure* on X is a finitely additive probability measure on $Def(X)$.
(iii) If μ is a Keisler measure on $Def_B(X)$ and $A \subseteq B$ we write $\mu|A$ for the restriction of μ to $Def_A(X)$.

Note that a complete type (of an element of X) over A is precisely a 0-1 valued Keisler measure on X over A.

For each L-formula $\phi(x,y)$ with x a variable of sort X, let S_ϕ be the sort whose elements are the subsets of X defined by instances of ϕ. So a global Keisler measure on X is given through a family $\{\mu_\phi : \phi(x,y) \in L\}$ of maps $\mu_\phi : S_\phi \to [0,1]$.

Keisler observes that any Keisler measure on X over A extends to a global Keisler measure on X. Moreover any Keisler measure on X over A extends to a unique countably additive measure on the σ-algebra generated by the A-definable subsets of X (see Theorem 1.2 in [13]). We will point out now a

way of extending a Keisler measure over a model to a global Keisler measure, as the construction will be useful later on.

Construction (*)*Let μ be a Keisler measure on X over a model M_0, viewed as a map from definable in M_0 subsets of $X(M_0)$ to $[0,1]$. Consider the structure $\langle M_0, [0,1], +, <, \mu_\phi \rangle_\phi$ consisting of M_0^{eq}, the real unit interval $[0,1]$, and for each ϕ, the map $\mu_\phi : S_\phi(M_0) \to [0,1]$ as well as the ordering and addition (modulo 1) on $[0,1]$. Take a saturated elementary extension $\langle M_0', [0,1]', +, <, \mu_\phi' \rangle_\phi$. Then the composition of μ' with the standard part map $st : [0,1]' \to [0,1]$ is a Keisler measure on X over M_0' extending μ. We may identify $\bar{M}$ with M_0'.*

One point of this construction is that the structure $\bar{M}$, equipped with the constructed measure, has some obvious "saturation" properties.

We have observed that a Keisler measure on X is (among other things) a sequence of maps from sorts S_ϕ to $[0,1]$. It would be natural to call μ *definable* if each $\mu_\phi : S_\phi \to [0,1]$ is definable in the sense of Definition 2.1. This is precisely (i) in the next definition.

Definition 2.5. Let μ be a (global) Keisler measure on X.
(i) Then μ is *definable over A* iff for each L-formula $\phi(x,y)$, and closed subset C of $[0,1]$, $\{b \in M : \mu(\phi(x,b)) \in C\}$ is type-definable over A.
Let M_0 be a small submodel of $\bar{M}$.
(ii) We say that μ is *finitely satisfiable* in M_0 if whenever $Y \subseteq X$ is definable and $\mu(Y) > 0$ then $Y \cap M_0 \neq \emptyset$.
(iii) We say that μ is *smooth over M_0* if μ is the unique (global) extension of $\mu|M_0$ to a measure on X. In this situation we also say that $\mu|M_0$ is smooth.

The notion of a smooth measure was also introduced by Keisler ([13]) although his definition is weaker than the above, for certain technical reasons. In any case, if μ is a $0-1$ measure given by a complete type then it is smooth if and only if the type is algebraic.

Here is a "nonalgebraic" example of a smooth Keisler measure: Let $\bar{M}$ be a saturated real closed field, and take X to be the interval $[0,1]$ in the sense of $\bar{M}$. The field of reals $\mathbb{R}$ is an elementary substructure of $\bar{M}$. The standard measure on the real unit interval $[0,1]^{\mathbb{R}}$ gives a Keisler measure on X over $\mathbb{R}$ which is easily seen to have a unique extension over $\bar{M}$. (This will be subsequently generalized in the last section.)

Lemma 2.6. *Let μ be a (global) Keisler measure on X. Suppose that μ is smooth over M_0. Then μ is both finitely satisfiable in M_0 and definable over M_0.*

Proof. Finite satisfiability is immediate from [13], Lemma 2.2 (which is itself based on Lemma 1.6 there), but for the sake of completeness we repeat the argument here.

It is clearly sufficient to prove that if X is a definable set in $\bar{M}$ with $\mu(X) > 0$ then it contains an M_0-definable Y with $\mu(Y) > 0$. Assume not, namely that all M_0-definable subsets of X have μ-measure zero. By the smoothness assumption, it is sufficient to show that there is *some* finitely additive Keisler-measure μ' on $\bar{M}$, extending $\mu|M_0$, with $\mu'(X) = 0$. By compactness, this amounts to showing, given finitely many M_0-definable sets $Y_1, \ldots, Y_k$, that there is a finitely additive probability measure μ' on the Boolean algebra generated by $Y_1, \ldots, Y_k, X$, which agrees with μ on the Y_i's. Let $\mathcal{B}_0$ be the Boolean algebra generated by the Y_i's. Without loss of generality, the Y_i's are atoms in $\mathcal{B}_0$ and hence each $Y_i \cap X$ is an atom in the Boolean algebra generated by $\mathcal{B}_0$ and X. We now let $\mu'(Y) = \mu(Y)$ for all $Y \in \mathcal{B}_0$ and $\mu'(Y_i \cap X) = 0$. This gives the desired measure μ' and proves that μ is finitely satisfiable.

The definability of μ over M_0 is more or less explained by a "Beth's Theorem for continuous logic". But we will be more direct. We make use of Construction (*) above. Consider the structure $\langle M_0, [0,1], +, <, \mu_\phi|M_0\rangle_\phi$ from there, equipped with constants for all elements (of M_0 and of the unit interval). Let T_1 be its theory. We saw that in a saturated model $\bar{M}_1$ of T_1, $\{st \circ \mu'_\phi : \phi \in L\}$ gives rise to a Keisler measure μ'' extending $\mu|M_0$. We may assume that $\bar{M}_1$ is an expansion of $\bar{M}$, and by the smoothness assumption, that $\mu'' = \mu$.

Fix an L-formula $\phi(x,y)$ where x is of sort X. Given a closed set $C \subseteq [0,1]$, we want to show that the set $X_1 = \{b : \mu(\phi(x,b)) \in C\}$ is type-definable in $\bar{M}$ over M_0. Note that the standard part map $st : [0,1]' \to [0,1]$ (where $[0,1]'$ is the unit interval in $\bar{M}_1$) is definable in $\bar{M}_1$ (over the empty set) in the sense of Definition 2.1, and by the definability of μ in $\bar{M}_1$, the set X_1 is type-definable over M_0 in $\bar{M}_1$, via a type $\Sigma(y)$.

Now, the smoothness assumption implies that $\Sigma(y)$ does not depend on the particular expansion $\bar{M}_1$ of $\bar{M}$. We can now apply the classical Beth Theorem (for types rather than formulas) and conclude that X_1 is type-definable in $\bar{M}$, over M_0. $\square$

Remark 2.7. *Let μ be a global Keisler measure on X. Let us define μ to be an heir of $\mu|M_0$ if for each L-formula $\phi(x,y)$ and $r \in [0,1)$, if for some $b \in \bar{M}$, $\mu(\phi(x,b)) > r$ then for some $b \in M_0$, $\mu(\phi(x,b)) > r$. Then the proof above shows that μ is the unique heir of $\mu|M_0$ over $\bar{M}$ if and only if μ is definable over M_0.*

The following relationship between Keisler measures and indiscernibles will be useful. It also appears in [14].

Lemma 2.8. *Let μ be a Keisler measure on X. Let x be a variable of sort X, let $\phi(x,y) \in L$, and let $\langle b_i : i < \omega \rangle$ be an indiscernible sequence such that for some $\epsilon > 0$, $\mu(\phi(x,b_i)) \geq \epsilon$ for all i. Then $\{\phi(x,b_i) : i < \omega\}$ is consistent.*

Proof. Let Y_{b_i} denote the set defined by $\phi(x,b_i)$. By construction (*) above and Ramsey's theorem, we may assume that the sequence $\langle b_i : i < \omega \rangle$ is also indiscernible with respect to the map μ, in particular that for each $i_1 < .. < i_n < \omega$ and $j_1 < .. < j_n < \omega$, $\mu(Y_{b_{i_1}} \cap ... \cap Y_{b_{i_n}}) = \mu(Y_{b_{j_1}} \cap ... \cap Y_{b_{j_n}}) = r_n$ say. So by assumption $r_1 > 0$.

Suppose for a contradiction that some finite intersection of the Y_{b_i}'s is empty. Choose maximal k such that $r_k > 0$. For $j \geq 0$ let $Z_j = Y_{b_1} \cap Y_{b_2} \cap .. \cap Y_{b_{k-1}} \cap Y_{b_{k+j}}$. Then each Z_j has measure $r_k > 0$ and their pairwise intersections have measure 0, a contradiction. ☐

3 NIP and some consequences

The definition of NIP (failure of independence property) was given in the Introduction. A well-known equivalence (see Theorem 12.17 of [25]) is:

Lemma 3.1. *T has the NIP if and only for any sequence $\langle b_i : i < \omega \rangle$ which is indiscernible over $\emptyset$ and formula $\phi(y)$, possibly with parameters, there is an i such that $\models \phi(b_j)$ for all $j > i$, or $\models \neg\phi(b_j)$ for all $j < i$.*

Notation: if $\phi(x), \psi(x)$ are formulas, let $\phi(x)\Delta\psi(x)$ denote the symmetric difference $(\phi(x) \wedge \neg\psi(x)) \vee (\neg\phi(x) \wedge \psi(x))$ of ϕ and ψ.

Corollary 3.2. *Suppose T has NIP. Let $\phi(x,y)$ be an L-formula, and $\langle b_i : i < \omega \rangle$ an indiscernible sequence. Then the set $\{\phi(x,b_{2j})\Delta\phi(x,b_{2j+1}) : j < \omega\}$ is inconsistent.*

Proof. Otherwise, let c realize $\{\phi(x, b_{2j})\Delta\phi(x, b_{2j+1}) : j < \omega\}$ and the formula $\phi(c, y)$ contradicts Lemma 3.1. $\square$

We now give some consequences of the NIP for Keisler measures. The main insight is due to Keisler ([13], Theorem 3.14). We are back to the context of $\bar{M}$ a saturated model of T and X a sort or $\emptyset$-definable set in $\bar{M}$.

Proposition 3.3. *Assume T has the NIP. Let μ be a (global) Keisler measure on X. Let $\phi(x, y)$ be a formula with x of sort X, and $\epsilon > 0$. Then there do not exist $\langle b_i : i < \omega\rangle$ such that $i \neq j$ implies $\mu(\phi(x, b_i)\Delta\phi(x, b_j) \geq \epsilon$.*

Proof. Suppose otherwise. Then by Construction (*) from Section 2, and Ramsey's theorem, we may assume in addition that $\langle b_i : i < \omega\rangle$ is indiscernible. By Lemma 2.8, $\{\phi(x, b_{2j})\Delta\phi(x, b_{2j+1}) : j < \omega\}$ is consistent, contradicting Corollary 3.2. $\square$

Corollary 3.4. *Assume T has NIP and let μ be a global Keisler measure on X. For definable subsets Y, Z of X, define $Y \sim_\mu Z$ if $\mu(Y\Delta Z) = 0$. Then there are only boundedly many $\sim_\mu$-classes of definable subsets of X. In particular there is a small model M_0 such that every definable subset Y of X is $\sim_\mu$ to some M_0-definable subset of X.*

Proof. If there are unboundedly many definable subsets of X modulo $\sim_\mu$ then we can clearly find a formula $\phi(x, y)$ and large set $\langle b_i : i \in I\rangle$ such that the measures of the pairwise symmetric differences of the $\phi(x, b_i)$ are > 0. By Construction (*) from Section 2, we may assume that $\langle b_i : i \in I\rangle$ is an indiscernible sequence with respect to μ as well, whereby $\mu(\phi(x, b_i)\Delta\phi(x, b_j)) \geq \epsilon$ for some fixed $\epsilon > 0$ and all $i \neq j$. This contradicts Proposition 3.3. $\square$

Our next result is in a somewhat different spirit.

Theorem 3.5. *Suppose T is countable with NIP. Let M_0 be a countable elementary substructure of $\bar{M}$. Let $p(x)$ be a complete 1-type over $\bar{M}$ which is finitely satisfiable in M_0. Let $U = \{X \cap M_0 : X \in p\}$. Then U is a Borel (in fact an F_σ) subset of the Polish space 2^{M_0}.*

Before going into the proof we give an easy example to illustrate the technique.

Remark 3.6. *Let T be countable, and let M_0 be a countable model. Then*
(i) The set $\{X \cap M_0 : X$ a definable subset of $\bar{M}\}$ is an F_σ (as a subset of

2^{M_0}).
(ii) Let $p(x) \in S_1(\bar{M})$ be definable. Then $\{X \cap M_0 : X \in p\}$ is an F_σ.

Proof. (i) Fix an L-formula $\phi(x, y)$, and $n < \omega$. Let $U_\phi = \{X \cap M_0 : X$ is defined by $\phi(x, c)$ for some $c\}$. By Example 2.2(ii), U_ϕ is closed. Then $U = \cup_\phi U_\phi$ is Borel and coincides with $\{X \cap M_0 : X$ definable subset of $\bar{M}\}$.
(ii) Suppose again $\phi(x, y) \in L$ and let $\psi(y, d)$ be a formula defining $p|\phi$. Then define U_ϕ just as above but requiring also that c realizes $\psi(y, d)$. □

The proof of Theorem 3.5 will go through several lemmas.

For now let T be an arbitrary complete theory with NIP.

Lemma 3.7. *For any $\phi(x, y) \in L$, there is some $N = N_\phi$, such that for any indiscernible sequence $\langle a_i : i < \omega \rangle$ and c, there do not exists $i_0 < i_1 < .. < i_N$ such that for each $j < N$, $\models \phi(a_{i_j}, c) \leftrightarrow \neg\phi(a_{i_{j+1}}, c)$.*

Proof. Otherwise, by compactness we find an indiscernible sequence $\langle a_i : i < \omega \rangle$ and c such that for each $i < \omega$, $\models \phi(a_i, c)$ iff $\models \neg\phi(a_{i+1}, c)$, contradicting Lemma 3.1. □

Recall that a type $p(x) \in S(\bar{M})$ is called *finitely satisfiable* in a model $M_0 \subseteq \bar{M}$ if every formula in $p(x)$ is satisfiable in M_0. If $p(x) \in S(\bar{M})$ is finitely satisfiable in a small model M_0, then we can build an indiscernible sequence $I = \langle a_0, a_1, ... \rangle$ over M_0 by letting a_0 realize $p|M_0$ and a_{i+1} realize $p|(M_0 \cup \{a_0, ..a_i\})$. Although the sequence I is not unique, its type $tp(\langle a_i : i < \omega \rangle / M_0)$ IS unique, and we call this type Q_{p,M_0}.

Let us now fix a type $p(x) \in S(\bar{M})$ which is finitely satisfiable in M_0, and let $Q = Q_{p,M_0}$. (So Q is a complete type over M_0 in variables $(x_i : i < \omega)$). Let Q_n be the restriction of Q to the variables $(x_0, .., x_n)$. Fix an L-formula $\phi(x, y)$ and some c from $\bar{M}$. We will say that a realization $(a_0, .., a_n)$ of Q_n is *good* for $\phi(x, c)$, if
(i) $\models \phi(a_i, c) \leftrightarrow \neg\phi(a_{i+1}, c)$ for all $i < n$, and
(ii) there does not exist a_{n+1} such that $(a_0, .., a_n, a_{n+1})$ realizes Q_{n+1} and $\models \phi(a_n, c) \leftrightarrow \neg\phi(a_{n+1}, c)$.

With this notation, we have the following:

Lemma 3.8. *For p as above, the following are equivalent:*
(i) $\phi(x, c) \in p$,
(ii) there is $k \le N_\phi$ and there is a realization $(a_0, .., a_k)$ of Q_k which is good for $\phi(x, c)$ such that $\models \phi(a_k, c)$,

Proof. Note first that by Lemma 3.7, for any c there is $k \leq N_\phi$ and realization $(a_0, .., a_k)$ of Q_k which is good for $\phi(x, c)$.
Now suppose $(a_0, .., a_k)$ realizes Q_k and is good for $\phi(x, c)$. Let M_1 be a small model containing $M_0 \cup \{a_0, ..a_k, c\}$ and let a realize $p|M_1$. Note that $(a_0, .., a_k, a)$ realizes Q_{k+1}. By the "goodness" of $(a_0, .., a_k)$ for $\phi(x, c)$, it follows that $\models \phi(a_k, c) \leftrightarrow \phi(a, c)$. But $\models \phi(a, c)$ iff $\phi(x, c) \in p$.

This is enough to prove the lemma. □

Let us now assume T and M_0 to be countable. We introduce some more notation: Fix k, and let $(Q_k^i : i < \omega)$ be an enumeration of the formulas in Q_k. Let $\psi_k^i(x_0, .., x_k, y)$ be the formula "$Q_k^i(x_0, .., x_k) \wedge \wedge_{j<k}(\phi(x_j, y) \leftrightarrow \neg\phi(x_{j+1}, y))$". Let $\chi_k^{j,i}(y)$ be:

$$\text{“}\exists x_0, .., x_k(\psi_k^j(x_0, .., x_k, y) \wedge (\neg\exists x_{k+1}(\psi_{k+1}^i(x_0, .., x_{k+1}))) \wedge \phi(x_{k+1}, y))''.$$

Corollary 3.9. *For any $c \in \bar{M}$, $\phi(x, c) \in p$ if and only if there is $k \leq N_\phi$ and there is $i < \omega$ such that c satisfies the formula $\chi_k^{j,i}(y)$ for all $j < \omega$.*

Proof. By Lemma 3.8 and the notation. □

Note that Corollary 3.9 gives us an F_σ-definition p over M_0.

In any case Theorem 3.5 follows from Corollary 3.9 as in the proofs of Remark 3.6. Note that the only real assumption on p we need is that it is finitely satisfiable in some small model (not necessarily M_0). □

4 Groups with finitely satisfiable generics

Here we introduce a certain desirable property of definable groups which we call fsg (standing for "finitely satisfiable generics") In Section 7 of the paper we prove that definably compact groups definable in o-minimal expansions of real closed fields have fsg.

Again we fix a saturated model $\bar{M}$ of T. G will denote a group, definable in $\bar{M}$ over $\emptyset$.

Definition 4.1. G has fsg (*finitely satisfiable generics*) if there is some global type $p(x)$ and some small model M_0 such that $p(x) \models$ "$x \in G$", and every left translate $gp = \{\phi(x) : \phi(g^{-1}x) \in p\}$ of p with $g \in G$, is finitely satisfiable in M_0.

The basic example of such a group is a *stable* group. (If G is stable, then there exists a global generic type p of G in the sense of stable group theory, namely every translate of p does not fork over $\emptyset$. But then by the characterization of forking in the stable context, every translate of p is finitely satisfiable in any submodel M_0.) In simple theories, however, definable groups will not, as a rule, have fsg. Also, the ordered group $\langle \mathbb{R}, <, + \rangle$ does not have fsg. On the other hand the *generically metastable* groups from [12] which were introduced in connection with definability in algebraically closed valued fields, *do* have fsg.

For the remainder of this paper we call a definable subset X of G (or the formula defining it) *left generic* if finitely many left translates of X cover G. Likewise with right generic. X is *generic* if it is both left and right generic. A partial type $\Sigma(x)$ implying $x \in G$ is left (right) generic if every formula in $\Sigma(x)$ is. Although this is accordance with established vocabulary in the case of stable theories, one should be aware that there is a discrepancy in the case of simple theories. Notice that if p is a global type in G, and X is a definable left generic subset of G then some left translate of X (i.e. of the formula "$x \in X$") is in p.

Proposition 4.2. *Suppose that G has fsg, witnessed by p and M_0, and let $X \subseteq G$ be definable. Then*
(i) X is left generic iff X is right generic (so we just say generic).
(ii) X is generic if and only if every left (right) translate of X meets M_0.
(iii) p is a generic type, as is any left or right translate of p.
(iv) If X is generic and $X = X_1 \cup X_2$ where the X_i are definable, then one of X_1, X_2 is generic.

Proof. Before we start let us note that
(*) $p^{-1} = \{\phi(x) : \phi(x^{-1}) \in p\}$ has the property that every right translate of it is finitely satisfiable in M_0
(i) Suppose X to be left generic. Then for any $c \in G$, cX is also left generic, so some left translate of cX is contained in p whereby cX is contained in some left translate gp of p. By the assumption (on p, M_0), cX meets M_0, namely there is $b \in G(M_0)$ such that $b \in cX$, so $c^{-1} \in Xb^{-1}$. We have shown that every element of G lies in Xb for some $b \in G(M_0)$. Compactness implies that finitely many right translates of X cover G, namely X is right generic. The other direction (right generic implies left generic) follows from (*) and symmetry.

(ii) follows from the proof of (i).
(iii) If X is in p then every left translate of X is in a left translate of p so meets M_0, whereby X is generic by (ii).
(iv) If X is generic, then X is in a translate of p. Thus one of X_1, X_2 is in the same translate of p. By (iii) one of X_1, X_2 is generic. □

Notice that Proposition 4.2 implies that G has fsg, witnessed by M_0, if and only if every definable generic subset of G meets M_0 and the complement of every nongeneric set is generic (the latter implies the existence of a generic type, while the first implies that a generic type is finitely satisfiable).

It follows from (iv) that, assuming that G has fsg, the set of nongeneric definable subsets of G forms an ideal $\mathcal{I}$ in the Boolean algebra of all definable subsets of G. So for a definable subset X of G, the stabilizer of X modulo this ideal, namely $Stab_{\mathcal{I}}(X) = \{g \in G : gX\Delta X \text{ is nongeneric}\}$ forms a subgroup of G. Note also that $Stab_{\mathcal{I}}(X)$ is type-definable (by countably many formulas). On the other hand for any global type q of G, $Stab(q)$ is defined to be the set of $g \in G$ such that $gq = q$. This is clearly a subgroup of G but on the face of it, has no definability properties.

Corollary 4.3. *Suppose that G has fsg. Then*
(i) There is a bounded number of (global) generic types,
(ii) G^{00} exists.
(iii) For each (global) generic type $p(x)$, $Stab(p) = G^{00} = \cap\{Stab_{\mathcal{I}}(X) : X \in p\}$.

Proof. (i) Each generic type is finitely satisfiable in M_0 by 4.2 (ii). So there are a bounded number of them. (Any global type p which is finitely satisfiable in a model M_0 is determined by $\{X \cap M_0 : X \in p\}$.)
(ii) Let (by part (i)) λ be the number of global generic types of G. Fix a generic type p. Let H be a type-definable subgroup of G of bounded index. So each coset of H is in a translate of p. The index of H in G is thus bounded by the number of (left) translates of p, which is at most λ. So we have an absolute bound (independent of the monster model) on the index of type-definable subgroups of G of bounded index, which clearly implies that G^{00} exists.
(iii) Fix a global generic type p of G. As G^{00} has bounded index some translate of G^{00} is in p (namely for some translate C of G^{00} $p(x)$ implies $x \in C$), whereby

(a) $Stab(p) \subseteq G^{00}$.
On the other hand clearly
(b) $\cap\{Stab_{\mathcal{I}}(X) : X \in p\} \subseteq Stab(p)$, as p only contains generic definable sets.
So to conclude the proof of (iii) it suffices to prove
(c) For each definable $X \in p$, $Stab_{\mathcal{I}}(X) \supseteq G^{00}$.
Suppose X is defined over a small model M containing M_0. Note that if $g, h \in G$ and $tp(g/M) = tp(h/M)$ then $gX \cap G(M) = hX \cap G(M)$, whereby $gX \Delta hX$ is not satisfiable in M_0 hence is nongeneric. It follows that the index of $Stab_{\mathcal{I}}(X)$ in G is bounded by the number of types over M, that is to say, $Stab_{\mathcal{I}}(X)$ has bounded index in G hence contains G^{00}. This proves (c) and completes the proof of the Corollary. □

Remark 4.4. *If G has fsg and M'_0 is any model then all generic definable sets meet $G(M'_0)$.*

Proof. Fix a formula $\phi(x, y)$ and $k < \omega$. By compactness there is a finite subset D of $G(M_0)$ such that if $X \subseteq G$ is defined by an instance of $\phi(x, y)$ and k left translates of X cover G, then X meets D. Let d be a finite tuple enumerating D. Then the above property of d can be expressed by a formula without parameters. As this formula is realized in every model M'_0 we are done. □

The following will be helpful in carrying out inductive proofs:

Proposition 4.5. *Let G be a $\emptyset$-definable group, and N a $\emptyset$-definable normal subgroup of G. Suppose that G/N and N both have fsg. Then so does G.*

Proof. Fix a small model M_0 witnessing that each of G/N and N have fsg. (In fact by 4.4 M_0 can be any submodel of $\bar{M}$.)

We will freely use Proposition 4.2, applied to each of G/N and N.

For X a definable subset of G, let us define Y_X to be $\{g/N \in G/N : g^{-1}X \cap N \text{ is generic in } N\}$. Now Y_X is *not* necessarily definable, so we cannot apply directly 4.2. But $Y_X = \cup_{i=1}^{\infty} Y_X^i$, where Y_X^i is the set of $g/N \in G/N$ such that i left translates of $g^{-1}X \cap N$ by elements of N cover N. Each Y_X^i *is* of course definable.

By compactness and the fact that G/N is fsg we have:
Claim 1. Finitely many left translates of Y_X cover G/N iff finitely many right translates of Y_X cover G/N iff for some $i < \omega$, Y_X^i is generic in G/N.

We will simply say "Y_X is generic in G/N" if the equivalent conditions of Claim 1 hold.

Claim 2. If Y_X is generic in G/N and $h \in G$ then each of Y_{hX} and Y_{Xh} are generic in G/N.

Proof. Just notice that $Y_X = (h/N)^{-1}Y_{hX} = Y_{Xh}(h/N)^{-1}$.

Claim 3. Suppose $X = X_1 \cup X_2$ where the X_i are definable. Then
(i) $Y_X = Y_{X_1} \cup Y_{X_2}$, and
(ii) If Y_X is generic in G/N then one of Y_{X_1}, Y_{X_2} is generic in G/N.

Proof. (i) As N has fsg, for each $g \in G$ $g^{-1}X \cap N$ is generic in N if and only if $g^{-1}X_1 \cap N$ or $g^{-1}X_2 \cap N$ is generic in N.

(ii) Assume that Y_X is generic in G/N, so there are $h_1, .., h_n \in G/N$ such that $\cup_{j=1,..,n} h_j Y_X = G/N$. By part (i) G/N is covered by the $h_j Y_{X_1}$ together with the $h_j Y_{X_2}$ for $j = 1, .., n$. Writing Y_{X_1} as $\cup_{i<\omega} Y^i_{X_1}$ and likewise for Y_{X_2} and applying compactness we see (as G/N has fsg) that either some $h_j Y^i_{X_1}$ is generic in G/N or some $h_j Y^i_{X_2}$ is generic in G/N. This suffices.

Claim 4. If Y_X is generic in G/N then $X \cap M_0 \neq \emptyset$.

Proof. By Claim 1, let i be such that Y^i_X is generic in G/N. Hence $Y^i_X \cap M_0 \neq \emptyset$. This means precisely that there is $h \in G(M_0)$ such that $h/N \in Y^i_X$. So $h^{-1}X \cap N$ is generic in N and $h \in G(M_0)$. Now, since N has fsg, the set $h^{-1}X \cap N$ contains an element of $G(M_0)$, which clearly implies that X does.

To conclude the proof, for X a definable subset of G, let us call X *-generic if Y_X is generic in G/N. By Claims 2 and 3, the family of *-generics is closed under (left or right) translation, and the family of non *-generics forms a proper ideal. Hence there is a global *-generic type p of G, and moreover by Claim 4, every translate of p is finitely satisfiable in M_0. This shows that G has fsg. □

Remark 4.6. *The* fsg *property can also be formulated in terms of measures. Namely, we say that a group G* has fsg_m *if there is a Keisler measure μ on G and there is some small model M_0 of $\bar{M}$ such that for every $g \in G$, the measure $g\mu$ (defined as $g\mu(X) = \mu(gX)$) is finitely satisfiable in M_0. It turns out that these formulations are equivalent:*

If G has fsg *then the generic type gives the desired 0-1 measure. For the converse, one goes through the proof of 4.2, using the fsg_m assumption instead of* fsg.

One could ask if the results of this section hold under the weaker assumption that there exists a type over a large saturated model, with a small

number of left translates. However variants of the following example will show that various lemmas can fail, even for rank one simple theories with a translation invariant type.

Example 4.7. *Let T_0 be the theory of vector spaces over $GF(2)$ with a symmetric irreflexive binary relation R (bearing no particular relation to $+$.) Let T be the model completion of T_0, $M \models T$. Let $C \subseteq M$ and $D \subseteq M \setminus \{0\}$ be arbitrary subsets. Then $\{R(x+b,c) : c \in C\} \cup \{\neg R(x+b,c) : c \notin C\} \cup \{R(x+c,x+b) : b-c \in D\} \cup \{\neg R(x+c,x+b) : b-c \notin D\}$ determines a complete type, which is M-translation invariant. But there are essentially no generic formulas. Theorem 3.5.*

5 Definably amenable groups

It is a convenient time to introduce the notion "definable amenability". Recall that an abstract (or discrete) group G is said to be *amenable* if there exists a left invariant finitely additive probability measure on the family of *all* subsets of G. Any solvable group is amenable.

Definition 5.1. *Let G be a definable group. We call G definably amenable if there is a left invariant Keisler measure on G.*

Remark 5.2. *(i) Any amenable group is definably amenable.*
(ii) Suppose T has a model M_0 such that G is defined over M_0 and $G(M_0)$ has a compact (Hausdorff) group topology such that every definable subset of G is Haar measurable. Then G is definably amenable.
(iii) If K is a (saturated) algebraically closed valued field and $n > 1$ then $SL(n,K)$ is not definably amenable.
(iv) If R is an expansion of a real closed field, then $PSL(2,R)$ is not definably amenable.
(v) $SO(3,\mathbb{R})$ is definably amenable, but not amenable as a pure group.

Proof. (ii) is proved by construction (*) applied to the (unique) normalized Haar measure on $G(M_0)$.
(iii) This follows by a similar proof to that in [12] showing that $SL(n,K)$ has no definable left generic type. (iv) Suppose μ_1 is a left invariant Keisler measure on $PSL(2,R)$. Recall the transitive action of $SL(2,R)$ on $\mathbb{P}^1(R) = R \cup \{\infty\}$. Define a Keisler measure μ on $\mathbb{P}^1(R)$ by $\mu(X) = \mu_1(\{g \in PSL(2,R) : g \cdot 0 \in X\}$. Then $\mu(hX) = \mu_1(\{g \in PSL(2,R) : h^{-1}g \cdot 0 \in X\} = \mu(X)$. But

let U be a small ball around 0; then using inversion we find gU, a ball around ∞; while using multiplication we can find hU such that $\mathbb{P}^1(R) = gU \cup hU$. So $\mu(U) \geq 1/2\mu(\mathbb{P}^1(R))$. This is true for an arbitrary small ball around any point, e.g. $0, 1, \infty$, giving $3/2 \leq 1$, a contradiction. (v) Every definable set is Lebesgue measurable. The pure group statement is due to Hausdorff, Banach and Tarski, see below.

Before continuing we take the opportunity to give a characterization of definable amenability (and the construction of an invariant Keisler measure on G from a suitable Grothendieck group of G). Fix a definable group G. By a nonnegative cycle in G we mean a "finite disjoint union" of definable subsets of G. Notationally consider a nonnegative cycle as $\{k_1X_1, .., k_nX_n\}$ where the k_i are nonnegative integers, and X_i are pairwise distinct definable subsets of G. There is the obvious notion of a map between two such cycles being definable, $1-1$ and given by piecewise left translations.

Definition 5.3. *A definable paradoxical decomposition of G is a definable $1-1$ piecewise translation from the disjoint union of G and Y to Y, for some nonnegative cycle Y.*

This is a variant of a notion due to Hausdorff. He actually considered a stronger notion, that would rule out the existence of any invariant finitely additive measure, not necessarily non-negative. His construction of a paradoxical decomposition of the two-dimensional sphere, or of $SO(3, \mathbb{R})$, completed by Banach and Tarski, requires the axiom of choice and is not represented in a definable way. On the other hand we have:

Proposition 5.4. *G is definably amenable if and only if G does not admit a definable paradoxical decomposition.*

Before entering the proof we introduce the relevant Grothendieck (semi-) group. Let $K_{semi}(G)$ be the semigroup whose elements are the nonnegative cycles $\sum_i k_iX_i$ in G modulo the equivalence relation of being in definable bijection by piecewise left translations. A typical element of $K_{semi}(G)$ can be written in the form $k_i[X]_{semi}$ where $[X]_{semi}$ is the class of the definable set X in $K_{semi}(G)$. Addition in the semigroup is the obvious thing.

Let us make a further identification: let $x_1, x_2 \in K_{semi}(G)$. Define $x_1 \sim_0 x_2$ if there is $y \in K_{semi}(G)$ such that $x_1 + y = x_2 + y$. Then the collection of $\sim_0$-classes, together with formal inverses, constitutes the Grothendieck group

$K_0(G)$. The class in $K_0(G)$ of a definable subset X of G is denoted $[X]_0$. (Likewise for a nonnegative cycle Y.)

Proof of Proposition 5.4. Suppose first that μ is a left invariant Keisler measure on G. Then a definable paradoxical decomposition could not exist since then we would have $\mu(G) + \mu(Y) \leq \mu(Y)$, contradicting $\mu(G) = 1$.
Conversely suppose G has no definable paradoxical decomposition. Let P_0 be the subsemigroup of $K_0(G)$ generated by the sets $[X]_0$ where X is definable.
Claim. $-n[G]_0 \notin P_0$ for all $n > 0$.
Proof. Otherwise $-n[G]_0 = [Y]_0$ (in $K_0(G)$) for some nonnegative cycle Y. But then $n[G]_0 + [Y]_0 = 0$ in $K_0(G)$, so $n[G]_{semi} + Y_{semi} + [Z]_{semi} = [Z]_{semi}$ in $K_{semi}(G)$ for some nonnegative cycle Z. But then clearly there is a definable injective piecewise translation map from the disjoint union of G and Z into Z, contradicting our assumption.

Let B be the tensor product of $\mathbb{Q}$ with $K_0(G)$, and

$$P = \{\alpha x : \alpha \in \mathbb{Q}, \alpha > 0, x \in P_0\}$$

By the Claim $-[G]_0 \notin P$. Let P' be a maximal subset of B containing P, closed under multiplication by positive rationals and addition, and such that $-[G]_0 \notin P'$. Define a partial ordering on B: $x \leq y \iff y - x \in P'$.

Claim. $\leq$ is a total ordering on B.
Proof. We have to show that for any $a \in B$, either $a \in P'$ or $-a \in P'$ If $a \notin P'$, let $P'' = \{x + \alpha a : x \in P', \alpha \in \mathbb{Q}, \alpha > 0\}$. Then by maximality $-[G]_0 \in P''$, i.e. $-[G]_0 = x + \alpha a, x \in P$, so $-a = \alpha^{-1}([G]_0 + x) \in P'$.

Now exists a unique order preserving semigroup homomorphism $h : B \to \mathbb{R}^{\geq 0}$ such that $[G]_0$ goes to 1. (Namely $h(b) = \alpha$ iff $[b : [G]_0] = [\alpha : 1]$ in the sense of [9] V. Def. 5, i.e. for all $m, n \in \mathbb{N}$, $mb < n[G_0]$ iff $m\alpha < n \cdot 1$). Let $\mu(X) = h([X]_0)$; this is clearly a left invariant Keisler measure on G.

Definable amenability of volumes. We will see in §8 that definably compact groups in o-minimal structures are definably amenable. The proof uses a deep structure theory for such groups. In the following paragraphs, not otherwise used in this paper, we consider a similar amenability property of definable compact definable sets. We do not know if this property holds in all o-minimal theories, but when it does we give a soft proof of definable amenability of definable compact groups. In particular this is valid for o-minimal expansions of RCF that are finitely satisfiable in expansions of $(\mathbb{R}, +, \cdot)$.

The proof actually yields more: that any definable group G, not necessarily definably compact, is *definably amenable for compact sets*. By definition this means: let $Def_{bdd}(G)$ be the family of definable subsets of definably compact subsets of G. Then for any $X \in Def_{bdd}(G)$ with nonempty interior, there exists a translation invariant finitely additive $\mu : Def_{bdd}(G) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ with $\mu(X) = 1$.

Let T be an o-minimal expansion of RCF, and fix $n \geq 1$. By "almost all' we will mean: away from a definable set of dimension $< n$. If $f : R^n \to R^n$ is definable, $|Jf|(c)$ denotes the absolute value of the determinant of the matrix of partial derivatives of ϕ at c; it exists almost everywhere. Let $V[n]$ be the set of bounded definable functions $R^n \to R^{\geq 0}$ with bounded support. By an *isomorphism* $\phi : f \to g$ we mean a definable bijection ϕ from a definable set containing the support of f to one containing the support of g, such that $f(x) = |J\phi|(x) \cdot g(\phi(x))$ almost everywhere. More generally define $f \sim g$ if one can write $f = \sum_{i=1}^{n} f_i, g = \sum_{i=1}^{n} g_i$ with f_i, g_i isomorphic. Let $[f]$ denote the $\sim$-class of f, and let $K_{semi}(V[n]) = \{[f] : f \in V[n]\}$. Define $[f] + [g] = [f + g]$. Let $K(V[n])$ be the corresponding group. We say that T is *definably amenable for volumes* if for each n and any $f \in V[n]$, either $f = 0$ a.e., or there exists an order-preserving semigroup homomorphism $I : K_{semi}(V[n]) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ with $0 < I(f) < \infty$.

Proposition 5.5. *Conditions (1) , (2) are equivalent. Also (3) or (4) imply (1) and (2); while (1) or (2) implies (5) and (5) implies (6).*

1. *T is amenable for volumes.*

2. *If $f, h \in V[n]$ and $f \sim f + h$, then $h = 0$ a.e.*

3. *Every finite $T_0 \subseteq T$ has a complete archimedean model.*

4. *T has definable primitives: for every definable function $f : R \to R$ in a model of T, there exists a definable function F such that almost everywhere $F' = f$.*

5. *Every definable group G of T is definably amenable for compact sets.*

6. *Every definably compact group in T is definably amenable.*

Proof. The proof of equivalence of (1) and (2) is identical to the proof of Proposition 5.4.

If (2) fails, then there exists a finite T_0 describing the situation. For instance if there exists an isomorphism $\phi : f \to f+h$, then ϕ is differentiable away from a set Y of dimension $n-1$; there exist $W_1, \ldots, W_k \subseteq R^{n-1}$ and continuously differentiable $e_i : W_i \to Y$ with $\cup_{i=1}^k e_i(W_i) = Y$; and $f+h = |J(\phi)| \cdot f \circ \phi$ away from Y. T_0 can state this, as well as the boundedness and piecewise differentiability of f, h, and that $h > 0$ on some open set. If T_0 has a complete archimedean model $N = (\mathbb{R}, +, \cdot, f^N, h^N, \ldots)$, then f^N, h^N are bounded integrable functions, and ϕ^N is C^1 away from Y^N, so by the change of variable formula $\int f^N = \int (f+h)^N$ (where $\int$ is the Lebesgue or Riemann integral.) But also $\int h^N > 0$, a contradiction. This shows that (3) implies (2). The proof that(4) implies (2) is similar: by a compactness argument, a function $f(x_1, \ldots, x_n)$ has a definable primitive $F_1(x_1, \ldots, x_n)$ with respect to the first variable, i.e. $\partial F_1 / \partial x_1 = f$ a.e. Now one can define integration using iterated integrals, and prove the change of variable formula and additivity using o-minimality. The proof of (3) implies (2) used no more than this.

(6) is obviously a special case of (5).

To prove (5) from (1) let $n = \dim(G)$. Fix an identification of some neighborhood of 1 in G with an open neighborhood of 0 in R^n. Let K_0 be the set of subsets Y of G contained in $b(int(U))$ for some injective continuously differentiable definable map $b : U \to G$, U a definably compact subset of R^n with interior $int(U)$. We begin by defining a map $\psi : K_0 \to K_{semi}(V[n])$.

For $g \in G$, let $T_g : G \to G$, $T_g(x) = g^{-1}x$. Given $Y \in K_0$, find a definably compact set $U \subseteq R^n$ and a definable injective C^1 map $b : U \to G$, with $Y \subseteq b(int(U))$. Let $f(x) = 0$ for $x \notin b^{-1}(Y)$, and for $x \in b^{-1}(Y)$ let $f(x) = |Jg|(x)$, where $g(y) = T_{b(x)} \circ b$. (Here we use the identification of a neighborhood of 1 with a neighborhood of 0 in $\mathbb{R}^n$; so $g : U \to \mathbb{R}^n$, and the Jacobian Jg is defined.) By continuity and definable compactness, f is bounded on U. If we pick a different $b' : U' \to Y$, with corresponding g', f', then $b' = b \circ e$ for some $e : U' \to U$ (defined on a neighborhood of the support of b), namely $e(u') = b^{-1}(b'(u'))$ on $(b')^{-1}((b(int(U))$. We have $g' = g \circ e$, $|Jg'|(x) = |Jg|(e(x))|Je|(x)$ so that f is isomorphic to f and $[f] = [f'] \in K_{semi}(V[n])$. Hence $[f]$ does not depend on the choice of (U, b) and we can define $\psi(Y) = [f]$.

Given $h \in G$, let $b'' = hb$; then $T_{b''(x)} \circ b'' = T_{b(x)} \circ b$, so $\psi(hU) = \psi(U)$. Thus ψ induces a well defined map $K_1 \to K_{semi}(V[n])$, where $K_1 = \{[Y] : Y \in K_0\} \subseteq K_{semi}(G)$. It is clear that $\psi(a+b) = \psi(a) + \psi(b)$ when $a, b, a+b \in K_1$, and that $a+b \in K_1$ implies $a \in K_1$. It follows that ψ extends

to homomorphism of ordered semigroups $\sum K_1 \to K_{semi}(V[n])$, where $\sum K_1$ is the semigroup generated by K_1.

According to [3], for any definably compact $Z \subseteq G$ there is a C^1 group manifold structure on G with finite chart $\{b_i : W_i \to G : i = 1, \ldots, r\}$ (with the W_i open subsets of R^n) and closed bounded $U_i \subseteq W_i$, such that $X \subseteq \cup_{i=1}^r b_i(int(U_i))$. Hence $\sum K_1 = K_{semi}(G)$.

If X has nonempty interior, then $\psi(X)$ cannot vanish almost everywhere, so by (1) there exists homomorphism $\mu : K_{semi}(V[n]) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ with $\mu(\psi(X)) = a > 0$. Now $(1/a)\mu \circ \psi$ demonstrates (5).

It is of course possible to combine (3) and (4), i.e. it suffices that every finite $T_0 \subseteq T$ be extendible to an *o*-minimal theory with definable primitives, or to one with an archimedean model.

Question 5.6. *Is every o-minimal theory amenable for volumes?*

6 Groups with NIP

Here we concentrate on definable groups in theories with NIP.

Suppose that μ is a Keisler measure on a definable group G. Then for any $g \in G$, we have another Keisler measure $g\mu$ on G, namely $g\mu(X) = \mu(gX)$. We say that μ is *left invariant* if $g\mu = \mu$ for all $g \in G$. Likewise for right invariant. The existence of a left invariant *type* of G is a very strong property. For example if G is stable, this implies that G is connected and the left invariant type is the unique generic type of G. However, even if there is NO invariant type, one may hope for there to exist an invariant measure.

The next proposition, due to Shelah [27], gives the existence of G^{00} for any definable (or even type-definable) group G in a theory with NIP. We had originally proved this under the additional assumption that G was definably amenable. In any case thanks to Shelah for allowing us to include the result and a proof.

Proposition 6.1. *Assume T has NIP. Let G be a definable group in $\bar{M}$, defined over $\emptyset$ say. Then G has a smallest type-definable subgroup of bounded index. If G^{00} is such then it is type-definable over $\emptyset$ and has index at most $2^{|T|}$.*

Proof. It is easy to see that any type-definable subgroup of G is the intersection of a family of subgroups each of which is type-defined by countably many

formulas (see for example Remark 1.4(ii) in [4]). So it suffices to prove that any subgroup H of G which is type-defined by countably many formulas and has bounded index in G has only a bounded number of distinct conjugates (under automorphisms of the ambient structure). So let us suppose, for a contradiction, that a is a countable tuple, H_a is type-definable by a countable partial type $\Sigma(x, a)$ over a, H_a is a subgroup of bounded index in G, and that $\{H_{a'} : tp(a') = tp(a)\}$ is unbounded (where $H_{a'}$ is type-defined by $\Sigma(x, a'))$. So by Erdos-Rado we have some indiscernible sequence $\langle a_i : i < \omega\rangle$ of realizations of $p = tp(a)$ such that $H_{a_i} \neq H_{a_j}$ for $i \neq j$.

Claim 1. Fix $i_0 < \omega$. Then $\cap\{H_{a_j} : j < \omega, j \neq i_0\}$ is NOT contained in $H_{a_{i_0}}$.
Proof of claim 1. Suppose otherwise. We can "stretch" the indiscernible sequence $\langle a_i : i < \omega\rangle$ by inserting some $(b_\alpha : \alpha < \kappa)$ in place of a_{i_0} (for any κ). But then each H_{b_α} contains $\cap_{j\neq i_0} H_j$. But $\alpha \neq \beta$ implies $H_{b_\alpha} \neq H_{b_\beta}$. So for any κ we can find at least κ many distinct subgroups of G each of which containing $\cap_{j\neq i_0} H_j$. As the latter has bounded index in G, we get a contradiction, proving the claim.

The claim clearly applies also to any stretching $\langle a_\alpha\rangle$ of the indiscernible sequence $\langle a_i : i < \omega\rangle$. So for each α, let c_α be such that $c_\alpha \notin H_\alpha$ but $c_\alpha \in H_\beta$ for all $\beta \neq \alpha$. Again by Erdos-Rado we may assume that the sequence $\langle (a_\alpha, c_\alpha) : \alpha < \kappa\rangle$ is indiscernible.
We may assume $\Sigma(x, a) = \{\phi_n(x, a) : n < \omega\}$ where $n < m$ implies $\models \phi_m(x, a) \to \phi_n(x, a)$.

Claim 2. There is $n < \omega$ such that for any α and any $d_1, d_2 \in H_\alpha$,
$\models \neg\phi_n(d_1 \cdot c_\alpha \cdot d_2, a_\alpha)$.
Proof of claim 2. As $tp(a_\alpha, c_\alpha)$ does not depend on α, it is enough to prove it for a fixed α. As $c_\alpha \notin H_\alpha$ we have the implication:
$y_1, y_2 \in H_\alpha \models \vee_n \neg\phi_n(y_1 \cdot c_\alpha \cdot y_2, a_\alpha)$. Now apply compactness.

We may clearly assume $n = 0$ in Claim 2.

Claim 3 For each finite $w \subset \kappa$ there is d_w such that for all α, $\models \phi_0(d_w, a_\alpha)$ iff $\alpha \notin w$.
Proof of claim 3. Let d_w be the product of the c_β for $\beta \in w$. So if $\alpha \notin w$, then as $c_\beta \in H_\alpha$ for each $\beta \in w$, $d_w \in H_\alpha$ hence satisfies $\phi_0(x, a_\alpha)$. On the other hand if $\alpha \in w$ then we can write d_w as $d_1 \cdot c_\alpha \cdot d_2$ where $d_1, d_2 \in H_\alpha$ (by an argument as above). So then we apply Claim 2.

Claim 3 shows that T has the independence property, a contradiction. So

G^{00} exists. Its type-definability over $\emptyset$ follows by uniqueness (any type-definable set which is $\emptyset$-invariant is type-definable over $\emptyset$, by quantifying out the parameters and using saturation). The bound on the index is clear too. $\square$

The existence of G^{00} (for G a definable group in a saturated model of T) had been proved earlier in various special cases. For example for o-minimal theories in [4]. In fact the latter proved in addition that G/G^{00} is a compact Lie group. For groups definable over $\mathbb{Q}_p$ in a model of $Th((\mathbb{Q}_p)_{an})$ this was done in [16]. For groups definable in Pressburger Arithmetics, it follows from work of Onshuus [15].

Here is an application of Proposition 6.1. Let us fix a compact Hausdorff group $\langle G, \cdot, ...\rangle$ equipped with additional first order structure. We use the term G to also denote this structure. Let us assume that (i) $Th(G)$ has the NIP, (ii) any definable subset of G is Haar measurable (with respect to the unique normalized Haar measure on G), and (iii) there is a neighbourhood basis of the identity of G consisting of definable sets, say U_i for $i \in I$.

Let G^* be a saturated elementary extension of G. So $\cap_{i\in I} U_i^*$ is the group of "infinitesimals", denoted by $inf(G^*)$ of G^*, and the quotient group (with the logic topology) is precisely G. By Proposition 6.1, $(G^*)^{00}$ exists, and in fact we have:

CLAIM. $(G^*)^{00}$ is precisely the group $inf(G^*)$ of infinitesimals of G^{**}.

Proof. By 6.1, $(G^*)^{00}$ is type-definable over $\emptyset$. As we already know that $inf(G^*)$ is type-definable, and of bounded index, it suffices to prove that any subgroup H of G^* which is type-definable over G by a countable set of formulas, and has bounded index, contains $inf(G^*)$. Let H be such, and suppose $H = \cap_n X_n$, where X_n is definable over G. We may assume that $X_n^{-1} \cdot X_n \subseteq X_{n-1}$ for all $n > 0$. Fix n. As H has bounded index in G^*, finitely many translates of $X_n(G)$ cover G whereby the Haar measure of $X_n(G)$ is > 0.

It follows (cf. the chapter on convolutions in [11]) that $(X_n^{-1} \cdot X_n)(G)$ has interior in G, so $(X_{n-1}^{-1} \cdot X_{n-1})(G)$ contains an open neighbourhood of the identity of G. Thus $X_{n-2}(G)$ contains some U_j. Hence H contains $inf(G^*)$, and the claim is proved.

Now measures come back into the picture. The following was proved in the stable case in [14].

Proposition 6.2. *Suppose T has NIP, and G is a $\emptyset$-definable group in $\bar{M}$ with fsg. Then there is a left invariant Keisler measure μ on G, which is moreover finitely satisfiable in some small model M_0.*

Proof. We will use 4.2 and 4.3. Let us fix a global generic type p of G over $\bar{M}$, such that $p(x)$ implies $x \in G^{00}$. The measure we construct will depend on p. We will first prove the proposition in the case where T is countable. By Remark 4.4 let us fix a countable model M_0 such that all generic definable subsets of G meet $G(M_0)$. Let $\mathbf{m}$ be the (unique) normalized Haar measure on the compact group G/G^{00}.

Claim. Let $X \subseteq G$ be definable. Then
(i) For $g \in G$, whether or not $gX \in p$ depends only on the coset of g modulo G^{00}.
(ii) $\{g/G^{00} : gX \in p\}$ is a Borel subset of G/G^{00} (so is Haar measurable).
Proof. (i) follows because $Stab(p) = G^{00}$.
(ii): By Remark 4.4 let M_0 be a countable model such that X is over M_0 and all generic definable subsets of G meet M_0. In particular p is finitely satisfiable in M_0. By Corollary 3.9, there are partial types $\Psi_i(y)$ over M_0 for $i < \omega$ such that for $g \in G$, $gX \in p$ iff $\models \vee_{i<\omega}\Psi_i(g)$. Let C_i be the closed subset of G/G^{00} determined by $\Psi_i(y)$, namely the image of the solution set of Ψ_i under the natural map taking to G to G/G^{00}. Then by part (i) of the Claim, $\{g/G^{00} : gX \in p\}$ is precisely $\cup_i C_i$, hence Borel.

By the Claim, we can define $\mu_p(X) = \mathbf{m}(\{g/G^{00} : gX \in p\})$. Then μ_p is finitely additive. For left invariance: let $g' \in G$, then $\{g/G^{00} : g \in g'X\} = \{g/G^{00} : g \in X\}g'/G^{00}$, so by right invariance of $\mathbf{m}$, $\mu_p(g'hX) = \mu_p(X)$.

Finally let us note that $\mu_p(X) > 0$ if and only if X is generic. Right implies left is true by invariance of μ_p. But if X is nongeneric, then no translate of X is in p, so $\{g/G^{00} : gX \in p\}$ is empty, hence $\mu_p(X) = 0$.

As we already know that every generic definable subset of G meets $G(M_0)$ for some small model M_0 we obtain finite satisfiability of μ_p in M_0.

So we have proved the proposition when T is countable. For the general case: given a definable subset X of G, let L_0 be a countable sublanguage of L in which G and X are definable. Let p_0 be the reduct of p to L_0 and let G_0^{00} be the smallest L_0-type-definable subgroup of G of bounded index. Let f be the canonical surjective homomorphism from G/G^{00} to G/G_0^{00}. Clearly f is continuous. Let $U = \{g/G^{00} : gX \in p\}$ and $U_0 = \{g/G_0^{00} : gX \in p_0\}$. Then $U = f^{-1}(U_0)$. But by the Claim in the countable case, U_0 is Borel. Hence

U is also Borel. So the Claim holds in general, and as above we obtain our measure μ_p. $\square$

It is natural to ask whether the measure μ_p defined above indeed depends on the type p or not. This and related issues will be tackled in a subsequent paper.

Our final result of this section will provide in a sense the missing link in the proof of the o-minimal conjectures.

Proposition 6.3. *Suppose that T has NIP, and G is a group definable in $\bar{M}$ such that G is definably amenable, and the set $\mathcal{I}$ of non (left) generic definable subset of G forms an ideal. Then*
(i) there are only a bounded number of definable subsets of G modulo the equivalence relation $X \sim_{\mathcal{I}} Y$ ($X \Delta Y \in \mathcal{I}$).
(ii) for each definable left generic $X \subseteq G$, $Stab_{\mathcal{I}}(X)$ ($= \{g \in G : gX\Delta X$ is nongeneric$\}$) is a (type-definable) subgroup of bounded index.

Proof. Let μ be a left invariant Keisler measure on G. Note that if X is a left generic definable subset of G then $\mu(X) > 0$ (as finitely many left translates of X cover G and these have all the same μ-measure as X). So if there unboundedly many $\sim_{\mathcal{I}}$-classes there will also be unboundedly many $\sim_{\mu}$-classes, contradicting Corollary 3.4. This proves (i).
(ii) follows immediately. $\square$

7 Interlude: Ind-definable and locally compact groups

As one of the authors remarked “it seems a pity to lose $SL_2(\mathbb{R})$”. So we give the notion of an Ind-definable group, point out that quotienting by a type-definable normal subgroup of bounded index yields a locally compact group, and develop analogues of some of the results so far for Ind-definable groups. We also state an appropriate version of the o-minimal conjectures from [23]. In any case we will be brief.

We still work in a saturated model $\bar{M}$. Ind-definable stands for “inductive limit of definable sets”. For notational reasons we will take the index set to be $\mathbb{N}$. So an *Ind definable set* X will be by definition a sequence $(X_n : n \in \mathbb{N})$ of definable sets together with definable embeddings $f_n : X_n \to X_{n+1}$ for

$n \in N$. The points of X correspond to sequences $(x, f_{n_0}(x), f_{n_0+1}(f_{n_0}(x)), ...)$ for some $x \in X_{n_0}$ and $n_0 \in \mathbb{N}$. It is convenient to view the f_n as inclusion maps, and so X as the increasing union $\bigcup_n X_n$. There are natural notions of an Ind-definable relation on X and Ind-definable functions between Ind-definable sets. For example an Ind-definable function g between $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$ is a function from X to Y such that for every m, n $\{x \in X_m : g(x) \in Y_n\}$ is definable and the restriction of g to this set is definable. We also have the obvious notion of an Ind-definable set, function,.. being defined over a given set A of parameters.

Definition 7.1. An *Ind-definable group* G is something of the form $G = \bigcup_n G_n$ where G_n are definable sets, $m : G \times G \to G$ is a group operation and when restricted to $G_n \times G_n$ has values in G_{n+1} (and is definable), and inversion when restricted to G_n has values also in G_n.

We could also say that an Ind-definable group G is a group object in the category of Ind-definable sets, noting that up to isomorphism G has the explicit form given in Definition 7.1.

By a *definable* subset of G we mean a definable subset of some G_n. Likewise a complete type extending G will be "concentrate" on some G_n.

For various reasons we will assume that
(*) G_0 generates G as a group.

Examples A basic example we have in mind for an Ind-definable group is a subgroup of a definable group G that is generated by a definable set $G_0 \subseteq G$ (such groups were called in [20], "$\bigvee$-definable groups" and in [6] "locally definable"). Another is the universal cover of $\langle [0,1), +(mod1)\rangle$, obtained as an increasing union of intervals $[-,n,n]$ and the obvious group operation. The group of definable automorphisms of a definable group G, say in a countable language, can also be viewed as an Ind-definable group, where the G_n's in the definition are obtained via the various definable families of automorphisms of G. Finally, "an infinite dimensional" example is, for a definable group G, the increasing union of $G, G\times G, \ldots, G^n, \ldots$, with the group operation acting coordinate-wise (such spaces are called by A. Piekosz, in preliminary notes, "weakly definable spaces").

Here are some analogues of the basic notions:

Definition 7.2. Let $G = \cup_n G_n$ be an Ind-definable group.
(i) Let X be a definable subset of G (i.e. of some G_n). We call X *left generic*

in G if for each m finitely many left translates of X by elements of G cover G_m.
(ii) By *a type-definable subgroup of* G we mean a subgroup H of G which is at the same time a type-definable subset of some G_n.
(iii) By *a Keisler measure on* G (or on any Ind-definable set for that matter) we mean a finitely additive real-valued function μ on definable subsets of G, namely for every definable subset X of G, $\mu(X) \geq 0$, $\mu(\emptyset) = 0$ and if X, Y are disjoint definable subsets of G then $\mu(X \cup Y) = \mu(X) + \mu(Y)$. (But note we do not require there be a finite bound on the measures of definable sets).

Note a difference with the usual situation: If G is Ind-definable it may have NO type-definable subgroup of bounded index (because G itself is not type-definable). In any case if G has a smallest type-definable subgroup of bounded index we will call it G^{00} and say "G^{00} exists".

As in Section 4, we will say that the Ind-definable (over $\emptyset$) group $G = \bigcup_n G_n$ has *finitely satisfiable generics* if there is a global complete type $p(x)$ of G (namely $p(x) \rightarrow$ "$x \in G_n$" for some n) such that every left translate of p by an element of G is finitely satisfiable in some fixed small model M_0.

The material from Section 4 generalizes as follows:

Proposition 7.3. *Suppose the Ind-definable group G has fsg. Then*
(i) Any definable subset X of G is left generic iff right generic iff every left (right) translate meets M_0.
(ii) there is a complete global generic type of G (in fact p as in the definition of fsg will be such, as well as any translate of p).
(iii) If X is a definable subset of G which is generic in G and $X = X_1 \cup X_2$ with X_i definable, then X_1 or X_2 is generic in G.
(iv) There is a smallest subgroup of G which has bounded index and is invariant over some small set of parameters.
(v) G^{00} exists and equals $Stab(q)$ for each global generic type q of G. (Hence the cosets of G^{00} in G correspond to the translates of q.)

Proof. Let p be the type given by fsg. So for all sufficiently large n, "$x \in G_n$" $\in p$. Likewise any definable subset X of p is in G_n for sufficiently large n. So given a definable left generic set X, there is an n such that "$x \in G_n$" $\in p$ and $X \subseteq G_n$. So (as finitely many left translates of X cover G_n) some left translate of X is in p hence X is in some left translate of p, so X meets M_0. Likewise every left translate of X meets M_0. Now fix m. Then for

every $g \in G_m$, gX meets M_0. By compactness there are $g_1, .., g_k \in G(M_0)$ such that for every $g \in G_m$, gX contains one of the g_i. But then for every $g \in G_m$, $g^{-1} \in Xg_1 \cup .. \cup Xg_k$. As $G_m = G_m^{-1}$, we see that finitely many right translates of X cover G_m. As m was arbitrary we conclude that X is right generic. The rest of (i) follows by the same argumentation (noting that every right translate of p^{-1} is finitely satisfiable in M_0).
(ii) follows from (i).
(iii) Again we may suppose that $X \subseteq G_n$ and $p(x) \models$ "$x \in G_n$. So some translate of X is in p, so X is in a translate of p, so X_1 or X_2 is in the same translate of p so is generic.
At this point we see that the collection of nongeneric definable subsets of G is an ideal. Call this ideal $\mathcal{I}$.
(iv) Suppose H to be a subgroup of G of bounded index which is A-invariant for some small set A. Then our given global generic type p determines a coset of H in G and every other coset of H in G corresponds to a translate of p. So the number of translates of p bounds the index of H in G. Hence there is smallest such H (even as A varies).
(v) requires a little finesse. First let X be any definable subset of G. Let $Stab_{\mathcal{I}}(X)$ be as in Section 4, namely $\{g \in G : gX\Delta X$ is nongeneric in $G\}$. So $Stab_{\mathcal{I}}(X)$ is a subgroup of G, but on the face of it has no definability properties. But we DO know that $Stab_{\mathcal{I}}(X)$ is invariant over the parameters defining X, and also has bounded index in G (as generics meet M_0). Now fix a global generic type q. By what we have just said, together with (iv), $\cap\{Stab_{\mathcal{I}}(X) : X \in q\} = H$ say is a subgroup of bounded index invariant over some small set, and H is clearly contained in $Stab(q)$. But there is an n such that $q(x) \models$ "$x \in G_n$", and therefore, as above, G_n and every G_m, $m > n$, are generic in G.

Clearly $Stab(q)$ is a subgroup of G contained in G_{n+1}. Thus $H \subseteq G_{n+1}$. Now for $X \in q$, let $Stab_{\mathcal{I}}^{n+1}(X) = \{g \in G_{n+1} : gX\Delta X$is nongeneric$\}$. But this is clearly type-definable (as we only have to say that finitely many translates of $gX\Delta X$ do not cover G_{n+2}.) As $H = \cap\{Stab_{\mathcal{I}}^{n+1}(X) : X \in q\}$, it follows that H is type-definable. So we have constructed a type-definable subgroup of G of bounded index. By (iv) there is a smallest one, so G^{00} exists. As in the earlier proof, G^{00} must contain $Stab(q)$. So $G^{00} = H = Stab(q)$. $\square$

We can easily generalize Proposition 6.1 as well.

Proposition 7.4. *Assume that T has the NIP, and $G = \bigcup_n G_n$ is an Ind-*

definable group (Ind-definable over $\emptyset$). Suppose that G HAS a type-definable subgroup of bounded index. Then it has a smallest one, G^{00} which is moreover normal and type-definable over $\emptyset$.

Proof. Note by assumption (*) that if H is a type-definable subgroup of G, contained in G_n say, then G/H has bounded cardinality iff G_n/H does.

By our assumptions, without loss of generality there is a type-definable subgroup H of G of bounded index, which is contained in G_0. The proof of Proposition 6.1 goes through word for word to give a type-definable subgroup L_0 of G of bounded index which is smallest among those contained in G_0. Likewise for each n there is a type-definable subgroup L_n of G which is smallest among those contained in G_n. But then $L_n \subseteq L_0$ so L_n is contained in G_0 so $L_n = L_0$. Thus $L_0 = G^{00}$. It is clearly normal and type-definable over $\emptyset$. ☐

Lemma 7.5. *Assume that G is an Ind-definable group as above and that G^{00} is a minimal type-definable subgroup of bounded index. Let $\pi : G \to G/G^{00}$ be the projection map and set $Y \subseteq G/G^{00}$ to be closed if and only if for every n, $\pi^{-1}(Y) \cap G_n$ is type-definable. Then these closed sets generate a locally compact topology on G, making it into a topological group.*

The compact sets in G/G^{00} are those closed Y such that $\pi^{-1}(Y)$ is contained in G_n for some n.

Proof. Left to the reader. ☐

Remark 7.6. *In fact one can formulate the notion of a "type-definable" equivalence relation E on an Ind-definable set Y, and assuming boundedly many classes one can define the "logic topology" on Y/E which will be locally compact. As we will only require the group case as in 7.5, we leave details of the general case to the reader.*

Finally we generalize Proposition 6.2 to the Ind-definable setting. Recall first that a left Haar measure on a locally compact group G is a left invariant Borel measure μ on G such that $\mu(X)$ is finite for X compact and positive for X open (so may take value ∞ on some Borel sets). A left Haar measure exists and is unique up to multiplication by a positive real.

Proposition 7.7. *Let G be an Ind-definable group with finitely satisfiable generics. Assume T has NIP. Then there is a left invariant Keisler measure on G which is moreover finitely satisfiable in some small model.*

Proof. As in the proof of 6.2, we may assume T to be countable. Let $\mathbf{m}$ be a right Haar measure on the locally compact group G/G^{00}. Let $p(x)$ be a global generic type extending G^{00}. Without loss of generality G^{00} is contained in G_0.
We would like (as in the proof of Proposition 6.2) to define a left invariant Keisler measure μ_p on G by stipulating that for any definable subset X of G, $\mu_p(X) = \mathbf{m}(\{g/G^{00} : gX \in p\})$.

So fix a definable subset X of G. Assume $X \subseteq G_n$. As before, whether or not gX is in p depends only on g/G^{00}. So the main point is to see that $\{g/G^{00} : gX \in p\}$ is Borel and has finite $\mathcal{M}$-measure.

Note that if $gX \in p$ then $g \in G_{n+1}$ (as $g \in G_0{\cdot}G_n$) and so $gX \subseteq G_{n+2}$. We copy the proof of Proposition 6.2 but defining now U to be $\{Y \cap G_{n+2}(M_0) : Y \in p\}$, and concluding that $\{g/G^{00} : gX \in p\}$ is a Borel subset of the compact set G_{n+1}/G^{00} hence has finite $\mathbf{m}$-measure. So we can define μ_p. Left invariance, finite additivity, and finite satisfiability in M_0 are proved as before. □

We conclude this interlude with a result that appears at first sight close to the conjectures for compact groups, mentioned in the introduction.

Proposition 7.8. *Let $\bar{M}$ be a saturated o-minimal structure (expansion of a real closed field) and G a definably connected group definable in $\bar{M}$. Then: There is a definably compact neighbourhood of the identity $G_0 = G_0^{-1}$ such that putting $G_n = G_0 \cdot .. \cdot G_0$, and $G_\infty = \bigcup_n G_n$, then the Ind-definable group G_∞ has a unique smallest type-definable subgroup of bounded index G_∞^{00} and the quotient $L = G_\infty/G_\infty^{00}$ with the "logic topology" is a connected Lie group of the same dimension as the o-minimal dimension of G.*

Proof. We can identify some neighborhood of 1 in G with a neighborhood of 0 in R^n; write $*$ for multiplication in G. The only possible linear approximation to $x * y$ is $x + y$, by associativity and the existence of differentiable inverse. So letting $|x| = max|x_i|$, for any $C > 0$, for all sufficiently small $e > 0$, if $|x| \leq e$ and $|y| \leq e$ then

$$|x * y - (x + y)| \leq C|(x, y)| \tag{1}$$

Take C infinitesimal, and then e infinitesimal compared to it, and let $U = \{x : |x| \leq e\}$, $H = \{x : |x| \leq (1/n)e, n = 1, 2, ...\}$. Then by (1) it is clear that H is a type-definable normal subgroup. Let $G_0 = U \cup U^{-1}$ in the sense of

(*), so as to have it symmetric; Let G_∞ be the Ind-definable group generated by U, or equivalently by G_0. Modulo H, $*$ agrees with $+$ on U, indeed on G_∞. In particular $G_\infty/H \simeq \mathbb{R}^n$.

It remains only to show that H equals G^{00} precisely, i.e. that G_∞/G_∞^{00} cannot have dimension *bigger* than the *o*-minimal dimension of G. We postpone this to §10, see Corollary 10.10. $\square$

However, note that the locally compact quotient we obtained is abelian; it is indeed a locally compact manifestation of the Lie algebra of G. We feel that the canonical compact quotient of a definably compact group K reflects better the structure of K; for instance K/K^{00} is non-abelian if K is non-abelian. In the general case too, there should also be a locally compact quotient whose structure is close to that of G. We do not at the moment have a precise statement of this, either in the compact or in the locally compact cases.

Note that the adjoint action $G \times L \to L$ is definable, in the sense of §2.

8 Proof of the o-minimal conjectures

We now use some of the preceding results to complete the proof of the conjectures on definably compact definable groups in *o*-minimal structures from [23]. In fact we will prove a bit more, namely that such groups have fsg and therefore, by 6.2 are definably amenable. Our main result (stated in the language of Definition 2.1) is:

Theorem 8.1. *Let $\bar{M}$ be a saturated o-minimal expansion of a real closed field. Let G be a definably connected definably compact group definable in $\bar{M}$. Then*
(i) G has fsg.
(ii) There is a definable surjective homomorphism $\pi : G \to H$ from G to a compact Lie group H such that the Lie group dimension of H equals the o-minimal dimension of G, and moreover such that any definable homomorphism from G to a compact group factors through π.

Of course the H in part (ii) of the theorem is precisely G/G^{00} equipped with the logic topology. We know from [4] that G^{00} exists and G/G^{00} is, as a topological group, a compact connected Lie group. As discussed in [19] we may assume that G is a definable closed subset of some $\bar{M}^n$ and that the

group operation on G is continuous with respect to the induced topology on G.

We will prove Theorem 8.1 by proving it when G is commutative and when G is "semisimple", and then use Proposition 4.5 among other things to conclude the general case. For the rest of this section $\bar{M}$ is a saturated o-minimal expansion of a real closed field.

Lemma 8.2. *Theorem 8.1 is true when G is commutative.*

Proof. We use additive notation for G. We first prove (ii). T being o-minimal has NIP. Also as G is commutative it is amenable so in particular definably amenable. Also by [19] the family of nongeneric definable subsets of G forms an ideal $\mathcal{I}$. We can apply Proposition 6.3 to conclude that $Stab_{\mathcal{I}}(X)$ is a type-definable subgroup of G of bounded index for any definable subset X of G. It is explained in [19] how this implies (ii), but we briefly recall the argument. For each n, we can find a definable subset X_n of G such that the sets $X_n, X_n + c_1, .., X_n + c_r$ form a partition of G, where $0, c_1, .., c_r$ are the elements of order n in G. Then $Stab_{ng}(X_n)$ contains no n-torsion (except 0). So if we know that each $Stab_{ng}(X)$ has bounded index it will follow that G^{00} is contained in every $Stab_{ng}(X_n)$, hence has no torsion. As G^{00} is divisible (see [4]), it follows that G and G/G^{00} have isomorphic torsion.

By a theorem of Edmundo and Otero (see [7]), the torsion of G is isomorphic to the torsion of $(S^1)^{dim(G)}$. Hence the compact commutative Lie group G/G^{00} must also be $(S^1)^{dim(G)}$. So (ii) is proved.

Now for (i). Let $\sim_{\mathcal{I}}$ be the equivalence relation: "$X \Delta Y$ is nongeneric" on definable subsets of G. By Proposition 6.3(i) there are only boundedly many definable subsets of G up to $\sim_I$. (Note this already proves that G has a bounded number of generic types.) Thus there is a small model M_0 such that G is defined over M_0 and for every generic definable subset Y of G there is an M_0-definable subset X of G such that $Y \sim_I X$. To prove that G has fsg it is clearly enough (given the existence of generic types) to prove that every generic definable subset Y of G meets $G(M_0)$. So let $Y \subseteq G$ be definable and generic.

Claim 1. There exists a definable subset $Y' \subseteq Y$ which is closed (in G, so in $\bar{M}^n$) and still generic.

Proof. First, we may replace Y by its interior. Now, for every $\epsilon > 0$ we consider the set Y_ϵ of all $y \in Y$ whose distance from the frontier of Y is

greater than ϵ (in the sense of $\bar{M}^n$). Because the frontier of Y is not generic, there is some $\epsilon > 0$ for which Y_ϵ is generic, and we take it to be Y'.

So we may assume Y to be closed. Let X be an M_0-definable subset of G such that $Y \sim_{ng} X$. We may clearly assume X to be closed (as $cl(X) \setminus X$ is nongeneric). Hence $X \cap Y$ is closed. Let $Z = X \setminus Y$.

Claim 2. The set of M_0-conjugates of $X \cap Y$ is finitely consistent.
Proof. Otherwise (as X is M_0-definable) finitely many M_0-conjugates of Z cover X. But Z is nongeneric in G as is any M_0-conjugate of Z. So X is the union of finitely many nongenerics, while itself being generic. This is a contradiction.

By Claim 2 and Theorem 2.1 of [19] (which comes out of Dolich's work [5]), $X \cap Y$ meets M_0, as does Y. This completes the proof of (i) and of Lemma 8.2. □

Let G be definable in $\bar{M}$. We will say that G has *very good reduction* if it is definably isomorphic, in $\bar{M}$, to a group G_1 with the following property: There is a sublanguage L_0 of the language L of $\bar{M}$ which contains $+, \cdot$ and there is an elementary substructure M_0 of $\bar{M}|L_0$ whose underlying set is $\mathbb{R}$, and such that G_1 is definable by an L_0-formula with parameters from M_0, i.e. from $\mathbb{R}$. (But note that M_0 need not be expandable to an elementary substructure of $\bar{M}$.)
Remark This notion of very good reduction is related to, but not identical with the algebraic-geometric notion in the case of saturated real closed fields and the natural valuation. In any case it is important to note that even if R is a saturated real closed field, there will be definable groups, even real algebraic ones which do not have very good reduction in the model theoretic sense above. Indeed, as was shown in [21], if R is a sufficiently saturated real closed field then not all elliptic curves over $acl(R)$ are definably isomorphic to each other (as groups). In fact this remains true even in an expansion of R to a structure R_{an} elementarily equivalent to $\mathbb{R}_{an}$. Now, in $\mathbb{R}_{an}$ all definable compact abelian groups of fixed dimension (defined over $\mathbb{R}$) are definably isomorphic to each other, therefore, even in R_{an} not all elliptic curves over $acl(R)$ have very good reduction.

Lemma 8.3. *Theorem 8.1 holds when G has very good reduction.*

Proof. Part (ii) of the theorem is precisely Fact 4.1 of [19]. The fact that the nongeneric sets form an ideal was proved in [19], but this as well as the rest

of (i) follows directly from Proposition 4.6 in the same paper, (which itself depends on results of Berarducci and Otero [2]). More precisely (with above notation) 4.6 of [19] states among other things that if $X \subset G$ is definable (in $\bar{M}$) then X is left generic iff right generic iff X contains an open set which is L_0-definable over M_0. This on the one hand implies that there exists a complete generic type, and on the other hand that if we pick M_1 to be any elementary substructure of $\bar{M}$ which contains M_0 then any generic definable subset of X meets M_1. Thus G has fsg. □

PROOF OF THEOREM 8.1.
Let G be an arbitrary definable, definably connected, definably compact group in $\bar{M}$. We prove the theorem by induction on $dim(G)$. If G is "semisimple", namely has no proper connected infinite definable normal commutative subgroup, then by [17], G is an almost direct product of finitely many almost definably simple groups $G_1, .., G_k$. ("Almost definably simple" means that the group is noncommutative and the quotient by some finite normal subgroup is definably simple.) Now by [18] (see the proof of (2) $\Rightarrow$ (3) in Theorem 5.1 there), any definably simple group is definably isomorphic to some semialgebraic group defined over $\mathbb{R}$. In particular, a definably simple group has very good reduction. It easily follows from Lemma 8.3 that Theorem 8.1 holds for a semisimple G.

Thus we may assume that G has an infinite, definably connected normal commutative subgroup N. By 8.2, the theorem is true of N, so we may assume $N \neq G$. By induction, the theorem is true for G/N, so by Proposition 4.5, G has fsg.

All that is left to do is to prove that the dimension of the compact Lie group G/G^{00} equals the o-minimal dimension of G. Notice first that the image of G^{00} under the projection onto G/N is necessarily $(G/N)^{00}$ (on one hand this image contains $(G/N)^{00}$; on the other hand the pre-image of $(G/N)^{00}$ is of bounded index and therefore contains G^{00}). Thus, it suffices to show that $G^{00} \cap N = N^{00}$. By [4] it is enough to prove:

Claim. $G^{00} \cap N$ is torsion-free.
Proof. Fix n. Let us first choose a definable subset X of N such that N is the disjoint union of the translates of X by the distinct n-torsion points $1, g_1, .., g_r$ say of N. (As usual X is obtained by considering the surjective endomorphism $\pi : x \to nx$ of N with itself, which has finite kernel, and use the existence of definable Skolem functions.) Likewise, using definable

Skolem functions, we can find a definable subset D of G which meets every coset of N in G in a unique point. It follows that the definable sets XD, g_1XD,..,g_rXD are disjoint and cover G. By Corollary 4.3, G^{00} is contained in $Stab_{\mathcal{I}}(X)$, and clearly the latter does not contain any of $g_1, .., g_r$. As n was arbitrary, it follows that $G^{00} \cap N$ is torsion-free. This completes the proof of Theorem 8.1. $\square$

By Proposition 6.2 we conclude also:

Corollary 8.4. *Let G be a definably compact group definable in $\bar{M}$. Then G is definably amenable. In fact there is a left invariant Keisler measure on G which is finitely satisfiable in some small model.*

Remarks

1. In the very last step of the above proof we showed that, under the given assumptions, $G^{00} \cap N = N^{00}$. This is not true in general, even if we assume that G, G/N and N all have NIP and fsg. Indeed, consider the group $G = \langle \mathbb{C}, + \rangle \oplus S^1$ (S^1 the circle group), with predicates for S^1 and all its semialgebraic subsets (but not for $\mathbb{C}$!). We have $G^{00} = G$, but $(S^1)^{00}$ is nontrivial.
2. Our proof of Theorem 8.1 depends in a crucial manner on the result [7] describing the torsion in definably compact commutative groups, which itself relies on quite intricate tools from algebraic topology. It would be desirable to have a "direct" proof of the latter in the spirit of the current paper. In fact we do have a reasonably elementary proof of the *existence* of torsion points (in commutative definably compact definably connected groups), which we sketch here:
(i) Using definable compactness, find a definable $X \subset G$ such that both X and its complement X^c are generic (this can be done similarly to the proof of Claim 1 above),
(ii) It follows that $Stab_{ng}(X) \neq G$, and thus (as we saw that $Stab_{ng}(X)$ has bounded index), $G^{00} \neq G$.
(iii) Since G/G^{00} is a compact connected commutative nontrivial Lie group ([4]) it has torsion, and since G^{00} is divisible ([4]), G itself has torsion.

3. Notice that if a definable G in an o-minimal structure has fsg then it necessarily implies that G is definably compact. Indeed, if G were not definably compact then by [22], G has a definable one dimensional, ordered subgroup H. Let $D \subseteq G$ be a definable set containing one representative for each coset

of H, and let $I = (0, \infty) \subseteq H$. Then $D \cdot I$ is nongeneric in G and so is its complement, contradicting 4.2.
4. The proof of the o-minimal group conjecture that we give here depends in the ambient real closed field in two different ways. Firstly, in order to ensure that our group can be embedded as a topological group into some R^n (see a discussion in [19]. Secondly (and more substentially) the above count of torsion points, By Edmundo and Otero was only carried out for expansions of real closed fields. The conjecture was proved separately for groups definable in ordered vector spaces over division rings (see [15]. [8]).

9 Compact domination

The third author has mentioned in previous papers that the *o*-minimal conjectures (solved in the last section) have the heuristic content that the map $G \to G/G^{00}$ should be a kind of intrinsic "standard part map". It is reasonable to attempt to give some concrete mathematical meaning to this, namely to come up with a model theory of "standard-part-like" maps (in a tame context). So we introduce the notion "compact domination". It is analogous to "stable domination" from [12] which was introduced with algebraically closed valued fields as a central example. We relate compact domination to the existence and uniqueness (and smoothness) of suitable Keisler measures, and prove that in the cases we understand well (very good reduction and dimension 1) definably compact groups in *o*-minimal structures *are* compactly dominated (by G/G^{00}).

We begin by working in a saturated model $\bar{M}$ of an arbitrary theory. When we say compact we mean compact Hausdorff. G denotes a definable (or even type-definable) group. We use freely the notion from Section 2 of a *definable map* from X to a compact space.

Definition 9.1. (i) Suppose X is type-definable, $\pi : X \to C$ is a definable surjective map from X to a compact space C, and μ is a probability measure on C. We say that X *is compactly dominated by* (C, μ, π) if for any definable (that is relatively definable with parameters) subset Y of X, and for every $c \in C$ outside a set of μ measure zero, either $\pi^{-1}(c) \subseteq Y$ or $\pi^{-1}(c) \subseteq X \setminus Y$. Namely,

$$\mu(\{c \in C : \pi^{-1}(c) \cap Y \neq \emptyset \text{ and } \pi^{-1}(c) \cap (X \setminus Y) \neq \emptyset\}) = 0.$$

(ii) Let G be a type-definable group. We say that G *is compactly dominated as a group*, if G is compactly dominated over by $(H, \mathbf{m}, \pi)$ where H is a compact group, $\mathbf{m}$ is the unique normalized Haar measure on H and π is a group homomorphism.

Note that in (i) above the set $\{c \in C : \pi^{-1}(c) \cap Y \neq \emptyset \text{ and } \pi^{-1}(c) \cap (X \setminus Y) \neq \emptyset\}$ is a closed subset of C, hence measurable.

When we work with a definable group G, we always refer to compact domination in the group sense.

Question 9.2. *To what extent does the definition of compact domination depend on the choice of "measure zero" as the notion of "smallness" in C?*

It would be interesting to investigate other possibilities. Smallness notions based on Baire category or dimension are more natural since they depend only on the topology; but in the context of groups the Haar measure also depends only on the topology and group structure, and connects naturally to the topics discussed in this paper. It would be nice if for groups these notions turned out to be equivalent.

Let P be compactly dominated via $\pi : P \to C$, where P and π are (type-) defined over $\emptyset$. We will say "$\theta(x, b)$ holds for almost all $x \in P$" if $\mu(\pi(\{x : \neg\theta(x, b)\}) = 0$. We can write: $(d_P x)\theta(x, b)$ for this. Note that this gives an partial type: $\{b : (d_P x)\theta(x, b)\}$ is type-definable over $\emptyset$. Indeed let $\{W_i\}_{i \in I}$ be the set of all closed subsets of C of positive measure; then $\pi^{-1}(W_i) = \cap_j W_{ij}$ for some definable sets W_{ij}. Now $\neg(d_P x)\theta(x, b)$ iff $\mu(\pi(\theta(x, b))) > 0$ iff $\pi(\theta(x, b))$ contains a closed set W_i of measure > 0, iff for some i, j $\pi(\neg\theta(x, b))$ contains W_{ij}. The case of Baire category is similar.

This is again in analogy with the stably dominated case, where one obtains definable types.

One could ask to what extent C is determined by P? If P is compactly dominated via $\pi_i : P \to C_i$, there exist continous maps $f_1 : C_1' \to C_2$ and $f_2 : C_2' \to C_1$, where C_i' is a large subset of C_i, such that $f_1\pi_1 = f_2$ for all $x \in \pi_1^{-1}(C_1')$, and dually. However in general f_1, f_2 are not inverses of each other.

Proposition 9.3. *Suppose G is compactly dominated by (H, π). Then*
(i) G has finitely satisfiable generics, and
(ii) G^{00} exists and equals $Ker(\pi)$.

Proof. Let us assume that G is compactly dominated (over $\emptyset$ say) by the data. We go through various claims which eventually yield (i) and (ii). Y will denote a definable subset of G and $\pi'(Y) = \{h \in H : \pi^{-1}(h) \subseteq Y\}$.
Claim 1. $\pi'(Y) \subseteq \pi(Y)$, $\pi(Y)$ is closed and $\pi'(Y)$ is open.
Proof. Clear.

Claim 2. $\mathbf{m}(\pi(Y)) = \mathbf{m}(\pi'(Y))$.
Proof. Because by the definition of compact domination $\mathbf{m}(\pi(Y)\setminus\pi'(Y)) = 0$.

Claim 3. The following are equivalent:
(a) Y is left (right) generic,
(b) $\pi(Y)$ is left(right) generic,
(c) $\mathbf{m}(\pi(Y)) > 0$,
(d) $\pi'(Y)$ is nonempty.

Proof. (a) implies (b) implies (c) are clear. Suppose now that (c) holds. Then by Claim 2, $\mathbf{m}(\pi'(Y)) > 0$, so in particular (d) holds. Now assume (d). Then Y contains a coset of $Ker\pi$, which is type-definable of bounded index, and hence Y is left and right generic.

Claim 4. If $Y = Y_1 \cup Y_2$ (where the Y_i are definable) and Y is generic, then Y_1 or Y_2 is generic.

Proof. By Claim 3, $\mathbf{m}(\pi'(Y)) > 0$, but the compact domination assumption implies that $\mathbf{m}(\pi'(Y)) = \mathbf{m}(\pi'(Y_1)) + \mathbf{m}(\pi'(Y_2))$, so again by Claim 3 we are done.

Let M_0 be an elementary substructure of $\bar{M}$ containing representatives of each coset of G modulo $Ker(\pi)$.

Claim 5. If Y is generic in G then Y meets M_0.
Proof. By Claim 3, Y contains a whole coset of $Ker(\pi)$.

By Claim 4 there is a global generic type p of G. Every translate of p is also generic so by Claim 5 is finitely satisfiable in M_0. Thus G has fsg giving part (i).

Let $\mathcal{I}$ be the ideal of nongeneric definable sets (which exists by Claim 5.)
Claim 6. $Ker(\pi) \subseteq Stab_{\mathcal{I}}(Y)$.
Proof. Let $g \in Ker(\pi)$. Then $\pi(Y \Delta gY) \subseteq (\pi(Y)\setminus\pi'(Y)) \cup (\pi(gY)\setminus\pi'(gY))$. By Claim 2 the latter has Haar measure 0, hence by Claim 3, $Y\Delta gY$ is nongeneric.

By Corollary 4.3, G^{00} exists and equals the intersection of all $Stab_{\mathcal{I}}(Y)$. Since

$ker\pi$ has bounded index in G, by Claim 6, G^{00} equals $Ker(\pi)$. $\square$

Note that it follows that if G is $\emptyset$-definable and compactly dominated over some parameters then it is compactly dominated over any model (as G^{00} is type-definable over $\emptyset$). We now aim towards the appropriate analogue of "existence and uniqueness of Haar measure" for compactly dominated groups. We begin with a group-free version:

Proposition 9.4. *Let X be type-definable over $\emptyset$, and compactly dominated over $\emptyset$ by (C, μ, π). Then:*
(i) There is a unique Keisler measure μ' on X with the property that $\mu(D) = \mu'(\pi^{-1}(D))$ for any closed $D \subseteq C$.
(ii) The Keisler measure μ' from (i) is smooth (over $\emptyset$).

Proof. We first start with an explanation. Given a Keisler measure ν on a definable or type-definable set, we can uniquely extend ν to a countably additive measure on the σ-algebra whose underlying "closed" sets are the type-definable subsets of X. (This was discussed and referenced in section 2.) So, as $\pi^{-1}(C)$ is type-definable (over $\emptyset$), then in (i) $\mu'(\pi^{-1}(C))$ makes sense, for a Keisler measure μ'. In fact it is precisely the infimum of the $\mu'(Y)$ for $\emptyset$-definable Y containing $\pi^{-1}(C)$.

In any case, let us first show the existence of μ': For Y a (relatively) definable subset of X, put $\mu'(Y) = \mu(\pi(Y))$. Note that μ' DOES satisfy the condition in (i): for if $D \subseteq C$ is closed, and $Y = \pi^{-1}(D)$, then Y is type-definable so equals $\cap_i Y_i$ where Y_i are (relatively) definable subsets of X. Let $D_i = \pi(Y_i)$. Then D_i is closed in C and $\cap_i D_i = D$. We may assume that the family $(Y_i)_i$ is closed under finite intersections. It follows that $\mu(D) = inf\{\mu(D_i) : i \in I\} = inf\{\mu'(Y_i) : i \in I\} = \mu'(Y)$.

We must check finite additivity of μ'. But if Y_1, Y_2 are disjoint definable subsets of X, then (by compact domination) $\mu(\pi(Y_1) \cap \pi(Y_2)) = 0$, hence $\mu'(Y_1 \cup Y_2) = \mu'(Y_1) + \mu'(Y_2)$.

Now for uniqueness: Suppose μ'' is another Keisler measure on X such that $\mu(D) = \mu''(\pi^{-1}(D))$ for any closed $D \subseteq C$. Let Y be an arbitrary definable subset of X. Then, since $\pi^{-1}\pi'(Y) \subseteq Y \subseteq \pi^{-1}\pi(Y)$, we have

$$\mu(\pi'(Y)) = \mu''(\pi^{-1}\pi'(Y)) \leq \mu''(Y) \leq \mu''(\pi^{-1}\pi(Y)) = \mu(\pi(Y)).$$

But $\mu(\pi'(Y)) = \mu(\pi(Y))$, hence $\mu'(Y) = \mu''(Y)$. So we have proved (i).

Recall that the smoothness of μ' over $\emptyset$ means by definition that $\mu'|\emptyset$ has precisely one extension to a Keisler measure on (all definable subsets of) X. However, since $\mu'|\emptyset$ satisfies the assumptions of (i) it follows that it has a unique extension. □

Theorem 9.5. *Suppose G is compactly dominated. Then G has a unique left invariant Keisler measure, which is moreover right invariant and smooth.*

Proof. Let $\pi : G \to H = G/G^{00}$. As before $\mathbf{m}$ denotes the Haar measure on H.

Let μ' be as in Proposition 9.4 and its proof, namely for definable $X \subseteq G$, $\mu'(X)$ is by definition $\mathbf{m}(\pi(X))$. Note that μ' will be both left and right invariant, as $\mathbf{m}$ is. By Proposition 9.4 μ' is also smooth.

Now suppose μ'' is another left invariant Keisler measure on G. Let M_0 be a model over which π is definable. By [13], $\mu''|M_0$ extends uniquely to a countably additive measure on the σ-algebra of subsets of G generated by the M_0-type-definable sets. We still call this $\mu''|M_0$ and note it is left invariant. But then $\mu''|M_0$ induces a left invariant countably additive measure on H: namely for B a Borel subset of H, define its measure to be $\mu''(\pi^{-1}(B))$. By uniqueness of Haar measure, this latter measure has to agree with $\mathbf{m}$. Hence we have shown that $\mathbf{m}(C) = \mu''(\pi^{-1}(C))$. By Proposition 9.4 (i), $\mu'' = \mu'$. This completes the proof. □

10 *o*-minimality and compact domination

Let $\bar{M}$ denote now a saturated *o*-minimal expansion of an ordered divisible group R.

Beraducci and Otero, in their paper [2], prove in effect, (for *o*-minimal expansions of real closed fields) that the unit n-cube I^n in $\bar{M}$ is compactly dominated, with respect to the standard part map to $I^n(\mathbb{R})$ equipped with Lebesgue measure. This is not stated explicitly in their paper, but follows from it. In any case we give below another proof of this fact (omitting the real closed field assumption), using the following beautiful theorem of Baisalov and Poizat (Recall that *a weakly o-minimal structure* is an ordered structure in which every definable subset of the linear ordering is a finite union of convex sets):

Theorem([1]) *If the saturated o-minimal structure*

$\bar{M}$ is expanded by any number of convex subsets of $\bar{M}$ then the resulting structure is weakly o-minimal.

Some notation: We let $\mathbb{R}$ denote a fixed copy of the reals, which we may assume is a subgroup of R (in particular, we have a copy of $\mathbb{Q}$ in R). Let Fin denote the set of finite elements of R (i.e. absolute value less than n for some $n \in \mathbb{N}$) and Inf the set of infinitesimals of R (absolute value $< 1/n$ for all $n \in \mathbb{N}$). Let π denote the "standard part map" from Fin onto Fin/Inf. Since Fin/Inf is archimedean (and $\bar{M}$ saturated) we can identify Fin/Inf with $\mathbb{R}$.

Let $\langle \bar{M}, Fin, Inf \rangle$ be the structure $\bar{M}$ equipped with unary predicates for Fin and Inf. Then the quotient group Fin/Inf is interpretable in it, and π induces a canonical bijection $i : Fin/Inf \to \mathbb{R}$.

Definition 10.1. *By $\mathbb{R}_{ind}$ (standing for "$\mathbb{R}$ with the induced structure") we mean the structure whose universe is $\mathbb{R}$ and whose relations are precisely the images under i of subsets of $(Fin/Inf)^n$ which are definable (with parameters) in $(\bar{M}, Fin, Inf)$.*

Lemma 10.2. *$\mathbb{R}_{ind}$ is o-minimal (in fact is an o-minimal expansion of the ordered group of $\mathbb{R}$).*

Proof. It is clear that $<$ and the graphs of $+$ and $\cdot$ are among the basic relations on $\mathbb{R}_{ind}$.

By [1] the structure $\langle \bar{M}, Fin, Inf \rangle$ is weakly o-minimal. Let $X \subseteq \mathbb{R}$ be definable in $\mathbb{R}_{ind}$. Then clearly $\pi^{-1}(X)$ is definable in $\langle \bar{M}, Fin, Inf \rangle$, so is a finite union of convex sets. So X has finitely many connected components. Thus $\mathbb{R}_{ind}$ is o-minimal. $\square$

Lemma 10.3. *Let $X \subset Fin^n$ be definable in $\bar{M}$ with $dim(X) < n$. Then $dim(\pi(X)) < n$ (in the o-minimal structure $\mathbb{R}_{ind}$).*

Proof. The proof is by induction on n, and is immediate for $n = 1$. For an arbitrary n, we may assume by cell decomposition that X is the graph of a continuous definable function $f : C \to R$, where C is a definable open set in R^{n-1}. By o-minimality of $\mathbb{R}_{ind}$, if $dim(\pi(X)) = n$ then it must contain the closure of a subset $U \times (q_1, q_2)$, for U an open rectangular box of rational coordinates (which we may assume is contained in C) and $q_1, q_2 \in \mathbb{Q}$.

Consider an arbitrary $x \in U(R)$ and r a rational number in (q_1, q_2). By assumptions, there exist x_1, x_2 infinitesimally close to x such that $f(x_1), f(x_2)$ are infinitesimally close to q_1, q_2, respectively. But then, by continuity, there

exists an x' infinitesimally close to x such that $f(x') = r$. It follows that $\pi(\{x \in U(R) : f(x) = r\}) = U$, which by induction implies that the set $\{x \in U(R) : f(x) = r\}$ has an interior in R^{n-1}. This can be done for any rational $r \in (q_1, q_2)$, contradiction. $\square$

Theorem 10.4. *Let I^n be the unit n-cube in R^n, π the standard part map from I^n to $I^n(\mathbb{R})$, and μ the Lebesgue measure on $I^n(\mathbb{R})$. Then I^n is compactly dominated (in $\bar{M}$) by $(I^n(\mathbb{R}), \mu, \pi)$.*

Proof. Let $X \subseteq I^n$ be definable in $\bar{M}$. Let Y be the frontier of X (the set of x such that every neighbourhood of x contains points both in X and not in X). Then $dim(Y) < n$. So $dim(\pi(Y)) < n$ by Lemma 10.3. As $\pi(Y)$ is definable in the o-minimal structure $\mathbb{R}_{ind}$, it follows that the Lebesgue measure of $\pi(Y)$ is 0. Note also that $\pi(Y)$ is closed. For $c \in I^n(\mathbb{R})$, the type-definable set $\pi^{-1}(c)$ is *definably connected* (cannot be written as the union of two relatively open relatively definable subsets). So for $c \in I^n(\mathbb{R}) \setminus \pi(Y)$, either $\pi^{-1}(c)$ is contained in X or contained in the complement of X. This proves compact domination. $\square$

$\bar{M}$.

We are now in a position to state a rather finer version of the conjectures from [23]. As before π denotes the homomorphism from G onto G/G^{00} and $\mathbf{m}$ denotes Haar measure on G/G^{00}.

Compact Domination Conjecture. Any definably compact group G (definable in a saturated o-minimal expansion of a real closed field) is compactly dominated (by the compact Lie group G/G^{00}, with its Haar measure $\mathbf{m}$).

Note that, by 9.3, if G (definably compact in saturated o-minimal expansion of a real closed field) is compactly dominated by H, then H has to coincide with the compact Lie group G/G^{00}.

The following lemma allows us to reduce the Compact Domination Conjecture to a simpler statement.

Lemma 10.5. *Suppose G is definably compact with $dim(G) = n$, and suppose that whenever $Y \subseteq G$ is definable and $dim(Y) < n$, then $\boldsymbol{m}(\pi(Y)) = 0$. Then G is compactly dominated by G/G^{00}.*

Proof. Note that G here is equipped with its "definable topology". We make use of a key result from [4] which says that G^{00}, and each translate of it,

are definably connected. It follows that if $X \subseteq G$ is definable, and Y is the frontier of X in G (which has dimension $< n$) then for all $c \notin \pi(Y)$, $\pi^{-1}(c)$ is either contained in X or disjoint from X. Now, just like in the proof of 10.4, we obtain compact domination. $\square$

The above conjecture, if proven true, will resolve an intriguing open problem regarding the connection between generic sets and torsion points.

Proposition 10.6. *Assume that G a definable abelian group in $\bar{M}$ and that G is compactly dominated by G/G^{00} (with its Haar measure). Then every definable generic subset of G contains a torsion point. In particular, if $X \subseteq G$ is generic then there are finitely many torsion points $g_1, \ldots, g_k$ such that $G = \bigcup_i g_i X$.*

Proof. If $X \subseteq G$ is generic then, by Claims 1 and 3 in the proof of 9.3, $\pi'(X) = \{g/G^{00} : gG^{00} \subseteq X\}$ is open in G/G^{00} and therefore contains a torsion point. Since G^{00} is divisible and torsion-free, the coset gG^{00}, and therefore X, contain a torsion point. The rest easily follows. $\square$

There is very little we currently know about the consequences of the above proposition. Indeed, we don't even know that every large set (namely, the complement of a definable subset of G of small dimension) contains a torsion point.

Theorem 10.7. *Let G be a definably compact group definable in an o-minimal $\bar{M}$. Then G is compactly dominated in either of the cases*
(i) $\bar{M}$ expands a real closed field and G has very good reduction.
(ii) $dim(G) = 1$.

Proof. Case (i): We assume that there is a sublanguage L_0 of L such that G is defined in L_0 over the elementary substructure $M_0 = \langle \mathbb{R}, +, <, .. \rangle$ of $\bar{M}|L_0$. Assume $dim(G) = n$. Then G has a covering by finitely many charts $U_1, .., U_r$, each of which is definably homeomorphic via some f_i to an open definable subset V_i of I^n (all definable in L_0 over M_0). Let $\mathbb{R}_{ind}$ be as above. As was pointed out earlier, G^{00} is exactly the collection of all elements in G that are infinitesimally close to e. Thus we identify G/G^{00} with $G(\mathbb{R}_{ind})$. Suppose $Y \subseteq G$ is definable with $dim(Y) < n$. Then working in the charts and using 10.3 we see that $dim(\pi(Y)) < n$ in the o-minimal structure $\mathbb{R}_{ind}$. Then clearly $\mathbf{m}(\pi(Y)) = 0$. (For example, working in the charts the Lebesgue measure of $\pi(Y)) = 0$, so the Haar measure must be 0 too.) Now apply 10.5.

Case (ii). If $dim(G) = 1$ then any definable subset Y of G of dimension < 1 is finite, so $\pi(Y)$ is finite too hence has Haar measure 0. Again apply 10.5. □

Corollary 10.8. *Suppose G is as in Theorem 10.7 Then there is a unique invariant Keisler measure on G, which is moreover smooth.*

Proof. By 9.5 and 10.7. □

Finally we return to the promised completion of the proof of Proposition 7.8, this time as an illustration of the compact domination conjecture. Actually the dominating group is *locally compact* in this case; the modification of the definition is evident. We show initially that G_∞ is (locally) compactly dominated via $G_\infty \to G_\infty/H$; as a bi-product, this gives $H = G_\infty^{00}$.

Proposition 10.9. *Let G, H be as in Proposition 7.8. Then G_∞ is (locally) compactly dominated via $G_\infty \to G_\infty/H$.*

Proof. Let $U(y) = \{x : |x| \leq y\}$. So $U = U(e)$. Let $\widetilde{G} = \cup_{N \in \mathbb{N}} U(Ne)$. By (1) of 7.8, $G_\infty \subseteq \widetilde{G}$, and $*, +$ coincide on $\widetilde{G}$ up to H. In fact $\widetilde{G} = G_\infty$, since $\widetilde{G}/H = \mathbb{R}^n$ and U/H contains an open neighborhood of 0 in $\mathbb{R}^n$.

Since $*, +$ coincide on $\widetilde{G}$ up to H, the proposition reduces to the case $G = (R^n, +)$, where $(R, +)$ is the underlying additive group of the o-minimal structure. In this case, add predicates for both $\{x : (\exists N \in \mathbb{N})|x| \leq Ne\}$ and for $\{x : (\forall N \in \mathbb{N})|x| < e/N\}$, obtain weak-o-minimality of their quotient by [1], and proceed as in the proof of 10.4. □

Corollary 10.10. $G_\infty^{00} = H$

Proof. A generic set has generic image in G_∞/H, hence contains a non-small subset of G_∞/H, hence the pullback contains at least one full coset of H. Since G_∞/H is bounded, $G_\infty^{00} = H$. □

References

[1] Y. Baisalov and B. Poizat, Paires de structures *o*-minimales, JSL 63 (1998), 570-578.

[2] A. Berarducci and M. Otero, An additive measure in *o*-minimal expansions of fields, Quarterly Journal of Mathematics 55 (2004), 411-419.

[3] A. Berarducci and M. Otero, Intersection Theory for o-minimal structures, APAL 107 (2001) 87-119, Lemma 10.5

[4] A. Berarducci, M. Otero, Y. Peterzil, and A. Pillay, A descending chain condition for groups in *o*-minimal structures, to appear in Annals of Pure and Applied Logic.

[5] A. Dolich, Forking and independence in *o*-minimal theories, JSL 69 (2004), 215-240.

[6] M. Edmundo, Locally definable groups in *o*-minimal structures, to appear in J. of Algebra.

[7] M. Edmundo and M. Otero, Definably compact abelian groups, Journal of Math. Logic, 4 (2004), 163-180.

[8] P. Eleftheriou, Ph.D. thesis, U. of Notre Dame.

[9] Euclid, Elements, Book V, tr. T.L. Heath.

[10] R. Grossberg, J. Iovino, and O. Lessmann, A primer of simple theories, Archive Math. Logic 41 (2002), 541-580.

[11] E. Hewitt and K. Ross,*Abstract harmonic analysis. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol 115, 2, Springer-Verlag, Berlin, 1979.

[12] E. Hrushovski, Valued fields, metastable groups, draft, 2004.

[13] H. J. Keisler, Measures and Forking, Annals of Pure and Applied Logic 43 (1987), 119-169.

[14] L. Newelski and M. Petrykowski, Weak generic types and coverings of groups, preprint 2004.

[15] A. Onshuus, Groups definable in $(\mathbb{Z}, +, <)$, preprint 2005.

[16] A. Onshuus and A. Pillay, Definable groups and *p*-adic Lie groups, submitted to Journals of LMS.

[17] Y. Peterzil, A. Pillay and S. Starchenko, Definably simple groups in *o*-minimal structures, Trans. AMS, 352 (2000), 4397-4419.

[18] Y. Peterzil, A. Pillay and S. Starchenko, Linear groups definable in o-minimal structures, J. of Algebra 247, (2002), 1-23.

[19] Y. Peterzil and A. Pillay, Generic sets in definably compact groups, submitted to Fund. Math.

[20] Y. Peterzil and S. Starchenko, Definable homomorphisms of abelian groups in *o*-minimal structures, Annals of Pure and Applied Logic 101 (2000), 1-27.

[21] Y. Peterzil and S. Starchenko, Uniform definability of the Weierstrass $\wp$-functions and generalized tori of dimension one, Selecta Mathematica, New Series 10 (2004) 525-550.

[22] Y. Peterzil and C. Steinhorn, Definable compactness and definable subgroups of *o*-minimal groups, J. London Math. Soc 59 (1999), 769-786.

definable set, June 2005, preprint

[23] A. Pillay, Type-definability, compact Lie groups, and *o*-minimality, J. Math. Logic, 4 (2004), 147-162.

[24] A. Pillay, *Geometric Stability Theory*, OUP, 1996.

[25] B.Poizat, *A Course in Model Theory*, Springer-Verlag, 2000.

[26] S. Shelah, Classification theory for elementary classes with the dependence property - a modest beginning, Sci. Japonicae, 59 (2004), 265-316.

[27] S. Shelah, Minimal bounded index subgroup for dependent theories, to appear in Proceedings AMS.